

CONFORMALLY OSSERMAN MANIFOLDS

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Dedicated to the memory of Novica Blažić (1959 – 2005), a remarkable mathematician and a wonderful person.

ABSTRACT. An algebraic curvature tensor is called Osserman if the eigenvalues of the associated Jacobi operator are constant on the unit sphere. A Riemannian manifold is called conformally Osserman if its Weyl conformal curvature tensor at every point is Osserman. We prove that a conformally Osserman manifold of dimension $n \neq 3, 4, 16$ is locally conformally equivalent either to a Euclidean space or to a rank-one symmetric space.

1. INTRODUCTION

An *algebraic curvature tensor* \mathcal{R} on a Euclidean space \mathbb{R}^n is a $(3, 1)$ tensor having the same symmetries as the curvature tensor of a Riemannian manifold. For $X \in \mathbb{R}^n$, the *Jacobi operator* $\mathcal{R}_X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by $\mathcal{R}_X Y = \mathcal{R}(X, Y)X$. The Jacobi operator is symmetric and $\mathcal{R}_X X = 0$ for all $X \in \mathbb{R}^n$.

Definition 1. An algebraic curvature tensor \mathcal{R} is called *Osserman* if the eigenvalues of the Jacobi operator \mathcal{R}_X do not depend on the choice of a unit vector $X \in \mathbb{R}^n$.

One of the algebraic curvature tensors naturally associated to a Riemannian manifold (apart from the curvature tensor itself) is the Weyl conformal curvature tensor.

Definition 2. A Riemannian manifold is called (*pointwise*) *Osserman* if its curvature tensor at every point is Osserman. A Riemannian manifold is called *conformally Osserman* if its Weyl tensor at every point is Osserman.

It is well-known (and is easy to check directly) that a Riemannian space locally isometric to a Euclidean space or to a rank-one symmetric space is Osserman. The question of whether the converse is true (“every pointwise Osserman manifold is flat or locally rank-one symmetric”) is known as the *Osserman Conjecture* [Os]. The first result on the Osserman Conjecture (the affirmative answer for manifolds of dimension not divisible by 4) was published before the conjecture itself [Chi]. In the following almost two decades, the research in the area of Osserman and related classes of manifolds, both in the Riemannian and pseudo-Riemannian settings, was flourishing, with dozens of papers and at least three monographs having been published [G1, G2, GKV].

At present, the Osserman Conjecture is proved almost completely, with the only exception when the dimension of an Osserman manifold is 16 and one of the eigenvalues of the Jacobi operator has multiplicity 7 or 8 [N1, N2, N3, N4]. The main difficulty lies in the fact that the Cayley projective plane (and its hyperbolic dual) are Osserman, with the multiplicities of the eigenvalues of the Jacobi operator being exactly 7 and 8; moreover, the curvature tensor of the Cayley projective plane is *essentially* different from that of the other rank-one symmetric spaces, as it does not admit a Clifford structure (see Section 2 for details). This is the only known Osserman curvature tensor without a Clifford structure, and to prove the Osserman Conjecture in full it would be largely sufficient to show that there are no other exceptions.

The study of conformally Osserman manifolds was started in [BG1], and then continued in [BG2, BGNSi, G2, BGNSt]. Every Osserman manifold is conformally Osserman (which easily follows from the

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formula for the Weyl tensor and the fact that every Osserman manifold is Einstein), as also is every manifold locally conformally equivalent to an Osserman manifold.

Our main results is the following theorem.

Theorem 1. *A connected C^∞ Riemannian conformally Osserman manifold of dimension $n \neq 3, 4, 16$ is locally conformally equivalent to a Euclidean space or to a rank-one symmetric space.*

Theorem 3 answers, with three exceptions, the conjecture made in [BGNSi] (for conformally Osserman manifolds of dimension $n > 6$ not divisible by 4, this conjecture is proved in [BG1, Theorem 1.4]).

Note that the nature of the three excepted dimensions in Theorem 3 is different. In dimension three the Weyl tensor gives no information on a manifold at all. In dimension four, even a “genuine” pointwise Osserman manifold does not have to be locally symmetric (see [GSV, Corollary 2.7], [Ol], for the examples of “generalized complex space forms”). As it is proved in [Chi], the Osserman Conjecture is still true in dimension four, but in a more restrictive version: one requires the eigenvalues of the Jacobi operator to be constant on the whole unit tangent bundle (a Riemannian manifold with this property is called *globally Osserman*). One might wonder, whether the conformal counterpart of this result is true. The following elegant characterization in dimension four is obtained in [BG2]: a four-dimensional Riemannian manifold is conformally Osserman if and only if it is either self-dual or anti-self-dual.

In dimension 16, both the conformal and the original Osserman Conjecture remain open (for partial results, see [N3, N4] in the Riemannian case and Theorem 3 in Section 3 in the conformal case).

As a rather particular case of Theorem 1, we obtain the following analogue of the Weyl-Schouten Theorem for rank-one symmetric spaces: a Riemannian manifold of dimension greater than four having “the same” Weyl tensor as that of one of the complex/quaternionic projective spaces or their noncompact duals is locally conformally equivalent to that space. More precisely:

Theorem 2. *Let M_0^n denote one of the spaces $\mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{H}H^{n/4}$, and let W_0 be the Weyl tensor of M_0^n at some point $x_0 \in M_0^n$. Suppose that for every point x of a Riemannian manifold M^n , $n > 4$, there exists a linear isometry $\iota : T_x M^n \rightarrow T_{x_0} M_0^n$ which maps the Weyl tensor of M^n at x on a positive multiple of W_0 . Then M^n is locally conformally equivalent to M_0^n .*

For $M_0^n = \mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$ and $n > 6$, the claim follows from [BG1, Theorem 1.4]. The fact that the dimension $n = 16$ is not excluded (compared to Theorem 1) follows from Theorem 3 (see Section 3).

We explicitly require all the object (manifolds, metrics, vector and tensor fields) to be smooth (of class C^∞), although all the results remain valid for class C^k , with sufficiently large k .

The paper is organized as follows. In Section 2, we give a background on Osserman algebraic curvature tensors and on Clifford structures and prove some technical Lemmas. The proof of Theorem 1 is given in Section 3. Theorem 1 is deduced from a more general Theorem 3. We first prove the local version using the differential Bianchi identity, and then the global version by showing that the “algebraic type” of the Weyl tensor is the same at all the points of a connected conformally Osserman Riemannian manifold (in particular, a nonzero Osserman Weyl tensor cannot degenerate to zero).

2. ALGEBRAIC CURVATURE TENSORS WITH A CLIFFORD STRUCTURE

2.1. Clifford structure. The property of an algebraic curvature tensor \mathcal{R} to be Osserman is quite algebraically restrictive. In the most cases, such a tensor can be obtained by the following remarkable construction, suggested in [GSV], which generalizes the curvature tensors of the complex and the quaternionic projective spaces.

Definition 3. A *Clifford structure* $\text{Cliff}(\nu; J_1, \dots, J_\nu; \lambda_0, \eta_1, \dots, \eta_\nu)$ on a Euclidean space \mathbb{R}^n is a set of $\nu \geq 0$ anticommuting almost Hermitian structures J_i and $\nu + 1$ real numbers $\lambda_0, \eta_1, \dots, \eta_\nu$, with $\eta_i \neq 0$. An algebraic curvature tensor \mathcal{R} on \mathbb{R}^n has a *Clifford structure* $\text{Cliff}(\nu; J_1, \dots, J_\nu; \lambda_0, \eta_1, \dots, \eta_\nu)$ if

$$(1) \quad \mathcal{R}(X, Y)Z = \lambda_0(\langle X, Z \rangle Y - \langle Y, Z \rangle X) + \sum_{i=1}^{\nu} \eta_i(2\langle J_i X, Y \rangle J_i Z + \langle J_i Z, Y \rangle J_i X - \langle J_i Z, X \rangle J_i Y).$$

When it does not create ambiguity, we abbreviate $\text{Cliff}(\nu; J_1, \dots, J_\nu; \lambda_0, \eta_1, \dots, \eta_\nu)$ to just $\text{Cliff}(\nu)$.

Remark 1. As it follows from Definition 3, the operators J_i are skew-symmetric, orthogonal and satisfy the equations $\langle J_i X, J_j X \rangle = \delta_{ij} \|X\|^2$ and $J_i J_j + J_j J_i = -2\delta_{ij} \text{id}$, for all $i, j = 1, \dots, \nu$, and all $X \in \mathbb{R}^n$. This implies that every algebraic curvature tensor with a Clifford structure is Osserman, as by (1) the Jacobi operator has the form $\mathcal{R}_X Y = \lambda_0 (\|X\|^2 Y - \langle Y, X \rangle X) + \sum_{i=1}^{\nu} 3\eta_i \langle J_i X, Y \rangle J_i X$, so for a unit vector X , the eigenvalues of \mathcal{R}_X are λ_0 (of multiplicity $n-1-\nu$ provided $\nu < n-1$), 0 and $\lambda_0 + 3\eta_i$, $i = 1, \dots, \nu$.

The converse (“every Osserman algebraic curvature tensor has a Clifford structure”) is true in all the dimensions except for $n = 16$, and also in many cases when $n = 16$, as follows from [N3] (Proposition 1 and the second last paragraph of the proof of Theorem 1 and Theorem 2), [N2, Proposition 1] and [N4, Proposition 2.1]. The only known counterexample is the curvature tensor $R^{\mathcal{O}P^2}$ of the Cayley projective plane (more precisely, any algebraic curvature tensor of the form $\mathcal{R} = aR^{\mathcal{O}P^2} + bR^1$, where R^1 is the curvature tensor of the unit sphere $S^{16}(1)$ and $a \neq 0$).

A Clifford structure $\text{Cliff}(\nu)$ on the Euclidean space \mathbb{R}^n turns it into a Clifford module (we refer to [ABS, Part 1], [H, Chapter 11], [LM, Chapter 1] for standard facts on Clifford algebras and Clifford modules). Denote $\text{Cl}(\nu)$ a *Clifford algebra* on ν generators x_1, \dots, x_ν , an associative unital algebra over \mathbb{R} defined by the relations $x_i x_j + x_j x_i = -2\delta_{ij}$ (this condition determines $\text{Cl}(\nu)$ uniquely). The map $\sigma : \text{Cl}(\nu) \rightarrow \mathbb{R}^n$ defined on generators by $\sigma(x_i) = J_i$ (and $\sigma(1) = \text{id}$) is a representation of $\text{Cl}(\nu)$ on \mathbb{R}^n . As all the J_i 's are orthogonal and skew-symmetric, σ gives rise to an *orthogonal multiplication* defined as follows. In the Euclidean space \mathbb{R}^ν , fix an orthonormal basis e_1, \dots, e_ν . For every $u = \sum_{i=1}^{\nu} u_i e_i \in \mathbb{R}^\nu$ and every $X \in \mathbb{R}^n$, define

$$(2) \quad J_u X = \sum_{i=1}^{\nu} u_i J_i X$$

(when $u = e_i$, we abbreviate J_{e_i} to J_i). The map $J : \mathbb{R}^\nu \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by (2) is an orthogonal multiplication: $\|J_u X\|^2 = \|u\|^2 \|X\|^2$ (similarly, we can define an orthogonal multiplication $J : \mathbb{R}^{\nu+1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $J_u X = u_0 X + \sum_{i=1}^{\nu} u_i J_i X$, for $u = \sum_{i=0}^{\nu} u_i e_i \in \mathbb{R}^{\nu+1}$, where e_0, e_1, \dots, e_ν is an orthonormal basis for the Euclidean space $\mathbb{R}^{\nu+1}$). For $X \in \mathbb{R}^n$, denote

$$\mathcal{J}X = \text{Span}(J_1 X, \dots, J_\nu X), \quad \mathcal{I}X = \text{Span}(X, J_1 X, \dots, J_\nu X).$$

Later we will also use the complexified versions of these subspaces which we denote $\mathcal{J}_{\mathbb{C}}X$ and $\mathcal{I}_{\mathbb{C}}X$ respectively, for $X \in \mathbb{C}^n$.

If \mathbb{R}^n is a $\text{Cl}(\nu)$ -module (equivalently, if there exists an algebraic curvature tensor with a Clifford structure $\text{Cliff}(\nu)$ on \mathbb{R}^n), then (see, for instance, [H, Theorem 11.8.2])

$$(3) \quad \nu \leq 2^b + 8a - 1, \quad \text{where } n = 2^{4a+b} c, \text{ } c \text{ is odd, } 0 \leq b \leq 3.$$

From (3), we have the following inequalities.

Lemma 1. *Let \mathcal{R} be an algebraic curvature tensor with a Clifford structure $\text{Cliff}(\nu)$ on \mathbb{R}^n . Suppose $n \neq 2, 4, 8, 16$. Then*

- (i) $n \geq 3\nu + 3$, with the equality only when $n = 6$, $\nu = 1$, or $n = 12$, $\nu = 3$, or $n = 24$, $\nu = 7$.
- (ii) $n > 4\nu - 2$, except in the following cases: $n = 24$, $\nu = 7$ and $n = 32$, $\nu = 8$.
- (iii) there exists an integer l such that $\nu < 2^l < n$.

2.2. Clifford structures on \mathbb{R}^8 and the octonions. The proof of Theorem 1 in the “generic case” will rely upon the fact that ν is small relative to n (with the required estimates given in Lemma 1). However, in the case $n = 8$, the number ν can be as large as 7, according to (3). Consider this case in more detail. As it is shown in [N2], not only every Osserman algebraic curvature tensor \mathcal{R} on \mathbb{R}^8 has a Clifford structure, but also that Clifford can be taken of one of the two (mutually exclusive) forms: either \mathcal{R} has a $\text{Cliff}(3)$ -structure, with $J_1 J_2 = J_3$, or an existing $\text{Cliff}(\nu)$ -structure can be “complemented” to a $\text{Cliff}(7)$ -structure. More precisely:

Lemma 2.

1. Suppose \mathcal{R} is an algebraic curvature tensor on \mathbb{R}^8 having a Clifford structure $\text{Cliff}(\nu; J_1, \dots, J_\nu; \lambda_0, \eta_1, \dots, \eta_\nu)$. Then exactly one of the following two possibilities may occur: either \mathcal{R} has a Clifford structure $\text{Cliff}(3)$ with $J_1 J_2 = J_3$, or there exist $7 - \nu$ operators $J_{\nu+1}, \dots, J_7$ such that J_1, \dots, J_7 are

anticommuting almost Hermitian structures with $J_1 J_2 \dots J_7 = \text{id}_{\mathbb{R}^8}$ and \mathcal{R} has a Clifford structure $\text{Cliff}(7; J_1, \dots, J_7; \lambda_0 - 3\xi, \eta_1 + \xi, \dots, \eta_\nu + \xi, \xi, \dots, \xi)$, for any $\xi \neq -\eta_i, 0$.

2. Let \mathbb{O} be the octonion algebra with the inner product defined by $\|u\|^2 = uu^*$, where $*$ is the octonion conjugation, and let $\mathbb{O}' = 1^\perp$, the space of imaginary octonions. Then, in the second case in assertion 1, there exist linear isometries $\iota_1 : \mathbb{R}^8 \rightarrow \mathbb{O}$, $\iota_2 : \mathbb{R}^7 \rightarrow \mathbb{O}'$ such that the orthogonal multiplication (2) is given by $J_u X = \iota_1(X)\iota_2(u)$.

Proof. 1. This assertion is proved in [N2, Lemma 5]. The proof is based on the fact that every representation σ of $\text{Cl}(\nu)$ on \mathbb{R}^8 , except for the representations of $\text{Cl}(3)$ with $J_1 J_2 = \pm J_3$, is a restriction of a representation of $\text{Cl}(7)$ on \mathbb{R}^8 , to $\text{Cl}(\nu) \subset \text{Cl}(7)$. It follows that the almost Hermitian structures J_1, \dots, J_ν defined by σ can be complemented by almost Hermitian structures $J_{\nu+1}, \dots, J_7$ such that J_1, \dots, J_7 anticommute, and so \mathcal{R} can be written in the form (1), with a formal summation up to 7 on the right-hand side (but with $\eta_i = 0$ when $i = \nu + 1, \dots, 7$). To obtain a $\text{Cliff}(7)$ -structure for \mathcal{R} , according to Definition 3, we only need to make all the η_i 's nonzero. This can be done using the identity

$$(4) \quad \langle X, Z \rangle Y - \langle Y, Z \rangle X = \sum_{i=1}^7 \frac{1}{3} (2\langle J_i X, Y \rangle J_i Z + \langle J_i Z, Y \rangle J_i X - \langle J_i Z, X \rangle J_i Y)$$

(which is obtained from the polarized identity $\|X\|^2 Y - \langle X, Y \rangle X = \sum_{i=1}^7 \langle J_i X, Y \rangle J_i X$ which follows from the fact that, for $X \neq 0$, the vectors $\|X\|^{-1} X, \|X\|^{-1} J_1 X, \dots, \|X\|^{-1} J_7 X$ form an orthonormal basis for \mathbb{R}^8). Then by (1), \mathcal{R} has a Clifford structure $\text{Cliff}(7; J_1, \dots, J_7; \lambda_0 - 3\xi, \eta_1 + \xi, \dots, \eta_\nu + \xi, \xi, \dots, \xi)$, for any $\xi \neq -\eta_i, 0$.

2. This assertion is also proved in [N2] (see the beginning of Section 5.1). The proof is based on the following. There are two nonisomorphic representations of $\text{Cl}(7)$ on \mathbb{R}^8 . Identifying \mathbb{R}^8 with the octonion algebra \mathbb{O} via a linear isometry these representations are given by the orthogonal multiplications $J_u X = uX$ and $J_u X = Xu$ respectively [LM, § I.8]. As $(uX)^* = X^*u^* = -X^*u$ for all $u, X \in \mathbb{O}$, $u \perp 1$, the first representation is orthogonally equivalent to the second one, with the operators J_i replaced by $-J_i$. Since changing the signs of the J_i 's does not affect the form of the algebraic curvature tensor (1), we can always assume that a $\text{Cliff}(7)$ -structure for an algebraic curvature tensor on \mathbb{R}^8 is given by the orthogonal multiplication $J_u X = \iota_1(X)\iota_2(u)$. \square

In the proof of Theorem 1 for $n = 8$, we will usually identify \mathbb{R}^8 with \mathbb{O} and of \mathbb{R}^7 with \mathbb{O}' via some fixed linear isometries ι_1, ι_2 and simply write the orthogonal multiplication in the form

$$(5) \quad J_u X = Xu,$$

where $X \in \mathbb{R}^8 = \mathbb{O}$, $u \in \mathbb{O}'$. The proof of Theorem 1 for $n = 8$ extensively uses the computations in the octonion algebra \mathbb{O} (in particular, the standard identities like $a^* = 2\langle a, 1 \rangle 1 - a$, $\langle a, b \rangle = \langle a^*, b^* \rangle = \frac{1}{2}(a^*b + b^*a)$, $a(ab) = a^2b$, $\langle a, bc \rangle = \langle b^*a, c \rangle = \langle ac^*, b \rangle$, $(ab^*)c + (ac^*)b = 2\langle b, c \rangle a$, $\langle ab, ac \rangle = \langle ba, ca \rangle = \|a\|^2 \langle b, c \rangle$, for any $a, b, c \in \mathbb{O}$, and the similar ones, see e.g. [HL, Section IV]) and the fact that \mathbb{O} is a division algebra (in particular, any nonzero octonion is invertible: $a^{-1} = \|a\|^{-2}a^*$). We will also use the bioctonions $\mathbb{O} \otimes \mathbb{C}$, the algebra over the \mathbb{C} with the same multiplication table as that for \mathbb{O} . As all the above identities are polynomial, they still hold for bioctonions, with the complex inner product on \mathbb{C}^8 , the underlying linear space of $\mathbb{O} \otimes \mathbb{C}$. However, the bioctonion algebra is not a division algebra (and has zero-divisors: $(i1 + e_1)(i1 - e_1) = 0$).

2.3. Technical lemma. In the proof of Theorem 1, we will use the following lemma.

Lemma 3. Suppose that $n > 4$, and additionally, if $n = 8$, then $\nu \leq 3$, and if $n = 16$, then $\nu \leq 7$.

1. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a homogeneous polynomial map of degree m such that for all $X \in \mathbb{R}^n$, $F(X) \in \mathcal{J}X$ (respectively $F(X) \in \mathcal{I}X$). Then there exist homogeneous polynomials c_i , $i = 1, \dots, \nu$ (respectively $i = 0, 1, \dots, \nu$), of degree $m - 1$ such that $F(X) = \sum_{i=1}^\nu c_i(X)J_i X$ (respectively $F(X) = c_0(X)X + \sum_{i=1}^\nu c_i(X)J_i X$).

2. Let $1 \leq k \leq \nu$ and let a_j , $1 \leq j \leq \nu$, $j \neq k$, be $\nu - 1$ vectors in \mathbb{R}^n such that for all $Y \in \mathbb{R}^n$,

$$(6) \quad \sum_{j \neq k} (\langle a_j, J_k Y \rangle J_j Y + \langle a_j, Y \rangle J_k J_j Y) = 0.$$

Then either $a_j = 0$ for all $j \neq k$, or $\nu = 1$, or $\nu = 3$, $J_1 J_2 = \varepsilon J_3$, $\varepsilon = \pm 1$, and $a_j = J_j v$ for all $j \neq k$, where $v \neq 0$.

3. Suppose n and ν are arbitrary numbers satisfying (3). Let N^n be a smooth Riemannian manifold and let J_1, \dots, J_ν be anticommuting almost Hermitian structures on N^n . Suppose that for every nowhere vanishing smooth vector field X on N^n , the distribution $\mathcal{J}X = \text{Span}(J_1 X, \dots, J_\nu X)$ is smooth (that is, the ν -form $J_1 X \wedge \dots \wedge J_\nu X$ is smooth). Then for every $x \in N^n$, there exists a neighbourhood $\mathcal{U} = \mathcal{U}(x)$ and smooth anticommuting almost Hermitian structures $\tilde{J}_1, \dots, \tilde{J}_\nu$ on \mathcal{U} such that $\text{Span}(\tilde{J}_1 X, \dots, \tilde{J}_\nu X) = \text{Span}(J_1 X, \dots, J_\nu X)$, for any vector field X on \mathcal{U} .

Proof. 1. It is sufficient to prove the assertion for the case $F(X) \in \mathcal{I}X$.

As for every $X \neq 0$, the vectors $X, J_1 X, \dots, J_\nu X$ are orthogonal and have the same length $\|X\|$, we have $\|X\|^2 F(X) = f_0(X)X + \sum_{i=1}^\nu f_i(X)J_i X$, where $f_0(X) = \langle F(X), X \rangle$, $f_i(X) = \langle F(X), J_i X \rangle$ are homogeneous polynomials of degree $m+1$ of X (or possibly zeros). Taking the squared lengths of the both sides we get $\|X\|^2 \|F(X)\|^2 = f_0^2(X) + \sum_{i=1}^\nu f_i^2(X)$, so the sum of squares of $\nu+1$ polynomials $f_0(X), f_1(X), \dots, f_\nu(X)$ is divisible by $\|X\|^2$. Let for $X = (x_1, \dots, x_n)$, $(\|X\|^2)$ be the ideal of $\mathbb{R}[X]$ generated by $\|X\|^2 = \sum_j x_j^2$, and let $\mathbf{R} = \mathbb{R}[X]/(\|X\|^2)$. We have $\sum_{i=0}^\nu \hat{f}_i^2 = 0$, where \hat{f}_i is the image of f_i under the natural projection $\pi : \mathbb{R}[X] \rightarrow \mathbf{R}$. If at least one of the \hat{f}_i 's is nonzero (say the ν -th one), then $\sum_{i=0}^{\nu-1} (\hat{f}_i/\hat{f}_\nu)^2 = -1$ in \mathbb{F} , the field of fractions of the ring \mathbf{R} . The field \mathbb{F} is isomorphic to the field $\mathbb{L}_{n-1} = \mathbb{R}(x_1, \dots, x_{n-1}, \sqrt{-d})$, where $d = x_1^2 + \dots + x_{n-1}^2$ (an isomorphism from \mathbb{L}_{n-1} to \mathbb{F} is induced by the map $(a + b\sqrt{-d})/c \rightarrow (a + bx_n)/c$, with $a, b, c \in \mathbb{R}[x_1, \dots, x_{n-1}]$, $c \neq 0$). By [Pf, Theorem 3.1.4], the level of the field \mathbb{L}_{n-1} , the minimal number of elements whose sum of squares is -1 , is 2^l , where $2^l < n \leq 2^{l+1}$. It follows that in all the cases when $\nu < 2^l < n$ we arrive at a contradiction. This means that $\hat{f}_i = 0$, for all $i = 0, \dots, \nu$, so each of the f_i 's is divisible by $\|X\|^2$ in $\mathbb{R}[X]$, so $F(X) = (\|X\|^{-2} f_0(X))X + \sum_{i=1}^\nu (\|X\|^{-2} f_i(X))J_i X$, with all the nonzero coefficients on the right-hand side being homogeneous polynomials of degree $m-1$. The claim now follows from assertion (iii) of Lemma 1.

2. If $\nu = 1$, equation (6) is trivially satisfied. If $\nu = 2$, the claim immediately follows by taking the inner product of (6) with $J_1 J_2 Y$. If $\nu = 3$, let $k = 3$ (without loss of generality). Taking the inner product of (6) with $J_1 Y$ we obtain $\langle a_1, J_3 Y \rangle \|Y\|^2 = \langle a_j, Y \rangle \langle J_1 J_3 J_2 Y, Y \rangle$. It follows that the polynomial $\langle J_1 J_3 J_2 Y, Y \rangle$ is divisible by $\|Y\|^2$. As the operator $J_1 J_3 J_2$ is symmetric and orthogonal, it equals $\pm \text{id}$. Hence $J_1 J_2 = \varepsilon J_3$, $\varepsilon = \pm 1$. Then (6) takes the form $\langle a_1, J_3 Y \rangle J_1 Y + \langle a_2, J_3 Y \rangle J_2 Y + \varepsilon \langle a_1, Y \rangle J_2 Y - \varepsilon \langle a_2, Y \rangle J_1 Y = 0$, which is equivalent to $a_1 = -\varepsilon J_3 a_2$. Acting by J_1 on the both sides we obtain $J_1 a_1 = J_2 a_2$, so $a_j = J_j v$, with $v = -J_1 a_1 = -J_2 a_2$ (we can assume $v \neq 0$, as otherwise $a_j = 0$).

Now assume $\nu > 3$ and denote $L = \text{Span}(a_j)$. As it follows from (6), if $Y \perp L$, then $J_k Y \perp L$, so L is J_k -invariant. Polarizing (6) we obtain $\sum_{j \neq k} (\langle a_j, J_k X \rangle J_j Y + \langle a_j, X \rangle J_k J_j Y) + \sum_{j \neq k} (\langle a_j, J_k Y \rangle J_j X + \langle a_j, Y \rangle J_k J_j X) = 0$. It follows that for all $X \perp L$ and all $Y \in \mathbb{R}^n$, $\sum_{j \neq k} (\langle a_j, J_k Y \rangle J_j X + \langle a_j, Y \rangle J_k J_j X) = 0$, that is, $J_u(Y)X = J_k J_{v(Y)}X$, where $u(Y) = \sum_{j \neq k} \langle a_j, J_k Y \rangle e_j$, $v(Y) = \sum_{j \neq k} \langle a_j, Y \rangle e_j$. Note that $u(Y), v(Y) \perp e_k$. Now, fix an arbitrary $Y \in \mathbb{R}^n$ and choose a unit vector $w \perp u(Y), v(Y), e_k$ (this is possible, as $\nu > 3$). Then $J_w J_u(Y)X = J_w J_k J_{v(Y)}X$, so $\langle J_w J_k J_{v(Y)}X, X \rangle = 0$, for all $X \in L^\perp$. If $v(Y) \neq 0$, then the operator $\|v(Y)\|^{-1} J_w J_k J_{v(Y)}$ is symmetric and orthogonal, so the maximal dimension of its isotropic subspace is $\frac{1}{2}n < n - (\nu - 1) = \dim L^\perp$ (the inequality follows from assertion (ii) of Lemma 1), which is a contradiction. Hence $v(Y) = 0$ for all $Y \in \mathbb{R}^n$, so all the a_j 's are zeros.

3. We first prove the lemma assuming $2\nu \leq n$. In this case, the proof closely follows the arguments of the proof of [N1, Lemma 3.1].

Let $Y_0 \in T_x N^n$ be a unit vector. As $2\nu \leq n$, there exists a unit vector $E \in T_x N^n$ which is not in the range of the map $\Phi : S^{\nu-1} \times S^{\nu-1} \rightarrow S^{n-1}$ defined by $\Phi(u, v) = J_u J_v Y_0$. Then $\mathcal{J}E \cap \mathcal{J}Y_0 = 0$. It follows that on some neighbourhood \mathcal{U}' of x there exist smooth unit vector fields Y and E_n such that $E_n(x) = E$, $Y(x) = Y_0$ and $\mathcal{J}E_n \cap \mathcal{J}Y = 0$ at every point $y \in \mathcal{U}'$. By the assumption, the ν -dimensional distribution $\mathcal{J}E_n$ is smooth, so we can choose ν smooth orthonormal sections E_1, \dots, E_ν of it, and then define anticommuting almost Hermitian structures \tilde{J}_α on \mathcal{U}' by $\tilde{J}_\alpha E_n = E_\alpha$ (so that $\tilde{J}_\alpha = \sum_{\beta=1}^\nu a_{\alpha\beta} J_\beta$, where $(a_{\alpha\beta})$ is the $\nu \times \nu$ orthogonal matrix given by $a_{\alpha\beta} = \langle E_\alpha, J_\beta E_n \rangle$).

Let $E_{\nu+1}, \dots, E_{n-1}$ be orthonormal vector fields on \mathcal{U}' such that E_1, \dots, E_n is an orthonormal frame, and let, for a vector field X on \mathcal{U}' , $\tilde{J}X$ denote the $n \times \nu$ matrix whose column vectors are $\tilde{J}_1 X, \dots, \tilde{J}_\nu X$ relative to the frame E_1, \dots, E_n . Then $(\tilde{J}X)^t \tilde{J}X = \|X\|^2 I_\nu$ and all the $\nu \times \nu$ minors of the matrix $\tilde{J}X$ are smooth functions on \mathcal{U}' . Moreover, the entries of the matrices $\tilde{J}E_i$, $i = 1, \dots, n$, are the rearranged entries of the matrices \tilde{J}_α , $\alpha = 1, \dots, \nu$, relative to the basis $\{E_i\}$, so to prove that the \tilde{J}_α 's are smooth it suffices to show that all the entries of the matrices $\tilde{J}E_i$ are smooth (on a possibly smaller neighbourhood). Denote $\tilde{J}E_i = \begin{pmatrix} K_i \\ P_i \end{pmatrix}$, where K_i and P_i are $\nu \times \nu$ and $(n - \nu) \times \nu$ matrices-functions on \mathcal{U}' respectively (note that $\tilde{J}E_n = \begin{pmatrix} I_\nu \\ 0 \end{pmatrix}$). For an arbitrary $t \in \mathbb{R}$, all the $\nu \times \nu$ minors of the matrix $\tilde{J}(E_i + tE_n) = \begin{pmatrix} K_i + tI_\nu \\ P_i \end{pmatrix}$ are smooth. For every entry $(P_i)_{k\alpha}$, $k = \nu + 1, \dots, n$, $\alpha = 1, \dots, \nu$, the coefficient of $t^{\nu-1}$ in the $\nu \times \nu$ minor of $\tilde{J}(E_i + tE_n)$ consisting of $\nu - 1$ out of the first ν rows (omitting the α -th row) and the k -th row is $\pm(P_i)_{k\alpha}$, so all the entries of all the P_i 's are smooth.

For the vector field Y , constructed at the beginning of the proof, denote $\tilde{J}Y = \begin{pmatrix} K \\ P \end{pmatrix}$. As $P = \sum_{i=1}^n \langle Y, E_i \rangle P_i$, all the entries of P are smooth on \mathcal{U}' . Moreover, as $\mathcal{I}Y \cap \mathcal{I}E_n = 0$, the spans of the vector columns of the matrices $\tilde{J}Y$ and $\tilde{J}E_n = \begin{pmatrix} I_\nu \\ 0 \end{pmatrix}$ have trivial intersection, so $\text{rk } P = \nu$, at every point $y \in \mathcal{U}'$. Therefore we can choose the rows $\nu + 1 \leq b_1 < \dots < b_\nu \leq n$ of the matrix P at the point x such that the corresponding minor $P_{(b)} = P_{b_1 \dots b_\nu}$ is nonzero. Then the same minor $P_{(b)}$ is nonzero on a (possibly smaller) neighbourhood $\mathcal{U} \subset \mathcal{U}'$ of x . Taking all the $\nu \times \nu$ minors of $\tilde{J}Y$ consisting of $\nu - 1$ out of ν rows of $P_{(b)}$ and one row of K we obtain that all the entries of K are smooth on \mathcal{U} . Moreover, for an arbitrary $t \in \mathbb{R}$, all the $\nu \times \nu$ minors of the matrix $\tilde{J}(tE_i + Y) = \begin{pmatrix} tK_i + K \\ tP_i + P \end{pmatrix}$ are smooth. Computing the coefficient of t in all the $\nu \times \nu$ minors of $\tilde{J}(tE_i + Y)$ consisting of $\nu - 1$ out of ν rows of $(tP_i + P)_{(b)}$ and one row of $tK_i + K$ and using the fact that all the entries of K, P and P_i are smooth on \mathcal{U} we obtain that all the entries of K_i are also smooth on \mathcal{U} . Therefore all the entries of all the matrices $\tilde{J}E_i$ are smooth on \mathcal{U} , hence the anticommuting almost Hermitian structures \tilde{J}_α are also smooth on \mathcal{U} .

As ν and n must satisfy inequality (3) (hence the inequalities of Lemma 1), the above proof works in all the cases except for the following: $n = 4, \nu = 3$ and $n = 8, \nu = 5, 6, 7$. The case $n = 4, \nu = 3$ is easy: taking any smooth orthonormal frame E_i on a neighbourhood of x and defining $\tilde{J}_\alpha = \sum_{\beta=1}^3 a_{\alpha\beta} J_\beta$ (with the orthogonal 3×3 matrix $(a_{\alpha\beta})$ given by $a_{\alpha\beta} = \langle E_\alpha, J_\beta E_4 \rangle$) we obtain that all the entries of the \tilde{J}_α relative to the basis E_i are ± 1 and 0 .

The proof in the cases $n = 8, \nu = 5, 6, 7$ is based on the fact that any set of anticommuting almost Hermitian structures J_1, \dots, J_ν on \mathbb{R}^8 , except when $\nu = 3$ and $J_1 J_2 = \pm J_3$, can be complemented by almost Hermitian structures $J_{\nu+1}, \dots, J_7$ to a set J_1, \dots, J_7 of anticommuting almost Hermitian structures on \mathbb{R}^8 (assertion 1 of Lemma 2).

If $n = 8, \nu = 7$, choose an arbitrary smooth almost Hermitian structure J_7 on some neighbourhood \mathcal{U} of x and complement it by anticommuting almost Hermitian structures J_1, \dots, J_6 at every point of \mathcal{U} . Then $\text{Span}(J_1 X, \dots, J_6 X) = (\text{Span}(X, J_7 X))^\perp$ is a smooth distribution, for every smooth nowhere vanishing vector field X on \mathcal{U} . This reduces the case $n = 8, \nu = 7$ to the case $n = 8, \nu = 6$.

Let $n = 8, \nu = 6$, and let J_7 be an almost Hermitian structure complementing J_1, \dots, J_6 at every point $x \in N^n$. Using the first part of the proof (or the fact that $J_7 X$ spans the one-dimensional smooth distribution $(\text{Span}(J_1 X, \dots, J_6 X) \oplus \mathbb{R}X)^\perp$, for every nonvanishing smooth vector field X) we can assume that J_7 is smooth on a neighbourhood \mathcal{U} of $x \in N^n$. Choose a smooth orthonormal frame E_1, \dots, E_8 on (a possibly smaller neighbourhood) \mathcal{U} such that the matrix of J_7 relative to E_i is $\begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}$ and define the almost Hermitian structure \tilde{J}_6 on \mathcal{U} by $\tilde{J}_6 E_2 = E_1, \tilde{J}_6 E_4 = E_3, \tilde{J}_6 E_6 = -E_5, \tilde{J}_6 E_8 = -E_7$. Then J_7 and \tilde{J}_6 anticommute, hence we can complement them by almost Hermitian structures J'_1, \dots, J'_5 on \mathcal{U} in such a way that $J'_1, \dots, J'_5, \tilde{J}_6, J_7$ are anticommuting almost Hermitian structures. Moreover, as both J_7 and \tilde{J}_6 are smooth on \mathcal{U} , the five-dimensional distribution $\text{Span}(J'_1 X, \dots, J'_5 X) = (\text{Span}(X, J_7 X, \tilde{J}_6 X))^\perp$ is smooth, for every smooth nowhere vanishing vector field X on \mathcal{U} . This reduces the case $n = 8, \nu = 6$ to the case $n = 8, \nu = 5$. Indeed, if $\tilde{J}_1, \dots, \tilde{J}_5$ are smooth anticommuting almost Hermitian structures on \mathcal{U} such that $\text{Span}(\tilde{J}_1 X, \dots, \tilde{J}_5 X) = \text{Span}(J'_1 X, \dots, J'_5 X)$, for every vector field X , then $\tilde{J}_1, \dots, \tilde{J}_5, \tilde{J}_6$ are the required almost Hermitian structures, as $\text{Span}(\tilde{J}_1 X, \dots, \tilde{J}_6 X) = \text{Span}(J'_1 X, \dots, J'_5 X, \tilde{J}_6 X) =$

$(\text{Span}(X, J_7 X))^\perp = \text{Span}(J_1 X, \dots, J_6 X)$, for every vector field X on \mathcal{U} , and \tilde{J}_6 anticommutes with every \tilde{J}_α , $\alpha = 1, \dots, 5$, since it anticommutes with every J'_α , $\alpha = 1, \dots, 5$.

Let $n = 8$, $\nu = 5$, and let J_6, J_7 be anticommuting almost Hermitian structures complementing J_1, \dots, J_5 at every point $x \in N^n$. As $\text{Span}(J_6 X, J_7 X) = (\text{Span}(J_1 X, \dots, J_5 X))^\perp$, by the first part of the proof, we can choose such J_6 and J_7 to be smooth on a neighbourhood \mathcal{U} of $x \in N^n$. Choose a smooth orthonormal frame E_1, \dots, E_8 on (a possibly smaller neighbourhood) \mathcal{U} as follows. First choose an arbitrary smooth unit vector field E_1 on \mathcal{U} . The vector fields $J_6 E_1$ and $J_7 E_1$ are orthonormal; set $E_2 = -J_6 E_1$, $E_3 = -J_7 E_1$. The unit vector field $J_6 J_7 E_1$ is orthogonal to $E_1, J_6 E_1, J_7 E_1$; set $E_4 = -J_6 J_7 E_1$. Choose an arbitrary smooth unit section E_5 of the smooth distribution $(\text{Span}(E_1, E_2, E_3, E_4))^\perp$ on \mathcal{U} . That distribution is both J_6 - and J_7 -invariant, so we can set, similar to above, $E_6 = J_6 E_5$, $E_7 = J_7 E_5$, $E_8 = -J_6 J_7 E_5$. Now define the almost Hermitian structure \tilde{J}_5 on \mathcal{U} whose matrix relative to the frame E_i is $\begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}$. Then \tilde{J}_5, J_6, J_7 are anticommuting almost Hermitian structures on \mathcal{U} , with $\tilde{J}_5 J_6 \neq \pm J_7$, hence we can complement them by almost Hermitian structures J'_1, \dots, J'_4 on \mathcal{U} in such a way that $J'_1, \dots, J'_4, \tilde{J}_5, J_6, J_7$ are anticommuting almost Hermitian structures. Moreover, as \tilde{J}_5, J_6, J_7 are smooth on \mathcal{U} , the four-dimensional distribution $\text{Span}(J'_1 X, \dots, J'_4 X) = (\text{Span}(X, \tilde{J}_5 X, J_6 X, J_7 X))^\perp$ is smooth, for every smooth nowhere vanishing vector field X on \mathcal{U} . By the first part of the proof, we can find smooth anticommuting almost Hermitian structures $\tilde{J}_1, \dots, \tilde{J}_4$ on (a possibly smaller) neighbourhood \mathcal{U} such that $\text{Span}(\tilde{J}_1 X, \dots, \tilde{J}_4 X) = \text{Span}(J'_1 X, \dots, J'_4 X)$, for every vector field X . Then $\tilde{J}_1, \dots, \tilde{J}_4, \tilde{J}_5$ are the required almost Hermitian structures, as $\text{Span}(\tilde{J}_1 X, \dots, \tilde{J}_5 X) = \text{Span}(J'_1 X, \dots, J'_4 X, \tilde{J}_5 X) = (\text{Span}(X, J_6 X, J_7 X))^\perp = \text{Span}(J_1 X, \dots, J_5 X)$, for every vector field X on \mathcal{U} , and \tilde{J}_5 anticommutes with every \tilde{J}_α , $\alpha = 1, 2, 3, 4$, since it anticommutes with every J'_α , $\alpha = 1, 2, 3, 4$. \square

3. CONFORMALLY OSSERMAN MANIFOLDS. PROOF OF THEOREM 1

Let M^n , $n \neq 3, 4$, be a smooth conformally Osserman Riemannian manifold. If $n = 2$, the manifold is locally conformally flat, so we can assume that $n > 4$. Combining the results of [N3] (Proposition 1 and the second last paragraph of the proof of Theorem 1 and Theorem 2), [N2, Proposition 1] and [N4, Proposition 2.1] we obtain that the Weyl tensor of M^n has a Clifford structure, for all $n \neq 16$, and also for $n = 16$ provided the Jacobi operator W_X has an eigenvalue of multiplicity at least 9 (note that the Jacobi operator of any Osserman algebraic curvature tensor on \mathbb{R}^{16} has an eigenvalue of multiplicity at least 7, by topological reasons). In the latter case, W has a Clifford structure $\text{Cliff}(\nu)$, with $\nu \leq 6$, at every point on M^n .

To prove Theorem 1 it therefore suffices to prove the following theorem.

Theorem 3. *Let M^n be a connected smooth Riemannian manifold whose Weyl tensor at every point $x \in M^n$ has a Clifford structure $\text{Cliff}(\nu(x))$. Suppose that $n > 4$, and additionally that if $n = 16$, then $\nu(x) \leq 4$. Then there exists a space M_0^n from the list $\mathbb{R}^n, \mathbb{C}P^{n/2}, \mathbb{C}H^{n/2}, \mathbb{H}P^{n/4}, \mathbb{H}H^{n/4}$ (the Euclidean space and the rank-one symmetric spaces with their standard metrics) such that M^n is locally conformally equivalent to M_0^n .*

Note that by Theorem 3, every point of M^n has a neighbourhood conformally equivalent to a domain of the same “model space”. Also note that Theorem 3, in comparison to Theorem 1, says something also in the case $n = 16$.

We start with a brief informal sketch of the proof of Theorem 3. First of all, we show that the Clifford structure for the Weyl tensor can be chosen locally smooth on an open, dense subset $M' \subset M^n$ (see Lemma 4 for the precise statement). To simplify the form of the curvature tensor R of M^n , we combine the λ_0 -part of W (from (1)) with the difference $R - W$, so that R has the form (7) for some smooth symmetric operator field ρ , at every point of M' . The technical core of the proof is Lemma 5 and Lemma 6 which establish various identities for the covariant derivatives of ρ , the J_i 's and the η_i 's, using the differential Bianchi identity for the curvature tensor of the form (7). Lemma 6 treats the case $(n, \nu) = (8, 7)$ and uses the octonion arithmetic, and Lemma 5, all the other cases (and uses the fact that ν is small compared to n , see Lemma 1). It follows from the identities of Lemma 5 and Lemma 6 that, unless the Weyl tensor vanishes, the metric on M' can be locally changed to a conformal one whose curvature tensor again has the form (7), but with the two additional features: firstly, all the η_i 's

are locally constant, and secondly, ρ is a Codazzi tensor, that is, $(\nabla_X \rho)Y = (\nabla_Y \rho)X$. By the result of [DS], the exterior products of the eigenspaces of a symmetric Codazzi tensor are invariant under the curvature operator on the two-forms. Using that, we prove in Lemma 7 that ρ must be a multiple of the identity, so, by (7), M' is locally conformally equivalent to an Osserman manifold. The affirmative answer to the Osserman Conjecture in the cases for n and ν considered in Theorem 3 [N1, Theorem 1.2] implies that M' is locally conformally equivalent to one of the spaces listed in Theorem 3. This proves Theorem 3 at the “generic” points. To prove Theorem 3 globally, we first show (using Lemma 8) that M splits into a disjoint union of a closed subset M_0 , on which the Weyl tensor vanishes, and nonempty open connected subsets M_α , each of which is locally conformal to one of the rank-one symmetric spaces $\mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{H}H^{n/4}$. On every M_α , the conformal factor f is a well-defined positive smooth function. Assuming that there exists at least one M_α and that $M_0 \neq \emptyset$ we show that there exists a point $x_0 \in M_0$ on the boundary of a geodesic ball $B \subset M_\alpha$ such that both $f(x)$ and $\nabla f(x)$ tend to zero when $x \rightarrow x_0$, $x \in B$ (Lemma 9). Then the positive function $u = f^{(n-2)/4}$ satisfies elliptic equation (37) in B , with $\lim_{x \rightarrow x_0, x \in B} u(x) = 0$, hence by the boundary point theorem, the limiting value of the inner derivative of u at x_0 must be positive. This contradiction implies that either $M = M_0$ or $M = M_\alpha$.

Proof of Theorem 3. Let M^n , $n > 4$, be a connected smooth Riemannian manifold whose Weyl tensor at every point has a Clifford structure. Define the function $N : M^n \rightarrow \mathbb{N}$ as follows: for $x \in M^n$, $N(x)$ is the number of distinct eigenvalues of the Jacobi operator W_X associated to the Weyl tensor, where X is an arbitrary nonzero vector from $T_x M^n$. As the Weyl tensor is Osserman, the function $N(x)$ is well-defined. Moreover, as the set of symmetric operators having no more than N_0 distinct eigenvalues is closed in the linear space of symmetric operators on \mathbb{R}^n , the function $N(x)$ is lower semi-continuous (every subset $\{x : N(x) \leq N_0\}$ is closed in M^n). Let M' be the set of points where the function $N(x)$ is continuous. It is easy to see that M' is an open and dense (but possibly disconnected) subset of M^n . The following lemma shows that the Clifford structure for the Weyl tensor is locally smooth on every connected component of M' .

Lemma 4. *Let M^n , $n > 4$, be a smooth Riemannian manifold whose Weyl tensor has a Clifford structure at every point. If $n = 16$, we additionally require that at every point $x \in M^{16}$, the Weyl tensor has a Clifford structure $\text{Cliff}(\nu(x))$ with $\nu(x) \neq 8$.*

Let M' be the (open, dense) subset of M^n at the points of which the number of distinct eigenvalues of the Jacobi operator associated to the Weyl tensor of M^n is locally constant. Then for every $x \in M'$, there exists a neighbourhood $\mathcal{U} = \mathcal{U}(x)$, a number $\nu \geq 0$, smooth functions $\eta_1, \dots, \eta_\nu : \mathcal{U} \rightarrow \mathbb{R} \setminus \{0\}$, a smooth symmetric linear operator field ρ and smooth anticommuting almost Hermitian structures J_i , $i = 1, \dots, \nu$, on \mathcal{U} such that the curvature tensor of M^n has the form

$$(7) \quad R(X, Y)Z = \langle X, Z \rangle \rho Y + \langle \rho X, Z \rangle Y - \langle Y, Z \rangle \rho X - \langle \rho Y, Z \rangle X \\ + \sum_{i=1}^{\nu} \eta_i (2 \langle J_i X, Y \rangle J_i Z + \langle J_i Z, Y \rangle J_i X - \langle J_i Z, X \rangle J_i Y),$$

for all $y \in \mathcal{U}$ and $X, Y, Z \in T_y M^n$. Moreover, if $n = 8$, then the curvature tensor has the form (7) either with $\nu = 3$ and $J_1 J_2 = \pm J_3$, or with $\nu = 7$, for all $y \in \mathcal{U}$.

Proof. Let X be a smooth unit vector field on M^n . As the Weyl tensor W is a smooth Osserman algebraic curvature tensor, the characteristic polynomial of $W_{X|X^\perp}$ (of the restriction of the Jacobi operator W_X to the subspace X^\perp) does not depend on X and is a well-defined smooth map $p : M^n \rightarrow \mathbb{R}_{n-1}[t]$, $y \rightarrow p_y(t)$, where $\mathbb{R}_{n-1}[t]$ is the $(n-1)$ -dimensional affine space of polynomials of degree $n-1$ with the leading term $(-t)^{n-1}$. As all the roots of $p_y(t)$ are real and the number of different roots is constant on every connected component of M' , the eigenvalues $\mu_0, \mu_1, \dots, \mu_l$ of $W_{X|X^\perp}$ are smooth functions and their multiplicities m_0, m_1, \dots, m_l are constant, on every connected component of M' (we chose the labelling in such a way that $m_0 = \max(m_0, m_1, \dots, m_l)$).

First consider the case $n \neq 8$. The Weyl tensor has a Clifford structure given by (1) at every point of M' . By Lemma 1, for $n > 4$, $n \neq 8, 16$, $n-1-\nu > \nu$, for any Clifford structure on \mathbb{R}^n . By (3), for $n = 16$, $\nu \leq 8$, so by the assumption, the inequality $n-1-\nu > \nu$ also holds for $n = 16$. Then the biggest multiplicity of an eigenvalue of $W_{X|X^\perp}$ is $n-1-\nu$ (see Remark 1). So

the number $\nu = n - 1 - m_0$ is constant and the function $\lambda_0 = \mu_0$ is smooth on every connected component of M' . Moreover, for every smooth unit vector field X on M' and every $i = 1, \dots, l$, the μ_i -eigendistribution of $W_{X|X^\perp}$ is $\text{Span}_{j:\lambda_0+3\eta_j=\mu_i}(J_j X)$. As λ_0 and μ_i are smooth functions on every connected component of M' , η_j also is. Moreover, on every connected component of M' , every distribution $\text{Span}_{j:\lambda_0+3\eta_j=\mu_i}(J_j X)$ is smooth and has a constant dimension m_i , for any nowhere vanishing smooth vector field X . By assertion 3 of Lemma 3, there exists a neighbourhood $\mathcal{U}_i(x)$ and smooth anticommuting almost Hermitian structures \tilde{J}_j (for the j 's such that $\lambda_0 + 3\eta_j = \mu_i$) on $\mathcal{U}_i(x)$ such that $\text{Span}_{j:\lambda_0+3\eta_j=\mu_i}(J_j X) = \text{Span}_{j:\lambda_0+3\eta_j=\mu_i}(\tilde{J}_j X)$. Let \tilde{W} be the algebraic curvature tensor on $\mathcal{U} = \cap_{i=1}^l \mathcal{U}_i(x)$ with the Clifford structure $\text{Cliff}(\nu; \tilde{J}_1, \dots, \tilde{J}_\nu; \lambda_0, \eta_1, \dots, \eta_\nu)$. Then $\nu = n - 1 - m_0$ is constant and all the \tilde{J}_i, η_i and λ_0 are smooth on \mathcal{U} . Moreover, for every unit vector field X on \mathcal{U} , the Jacobi operators \tilde{W}_X and W_X have the same eigenvalues and eigenvectors by construction, hence $\tilde{W}_X = W_X$, which implies $\tilde{W} = W$.

Now consider the case $n = 8$. By Lemma 2, at every point $x \in M'$, the Weyl tensor either has a $\text{Cliff}(3)$ -structure, with $J_1 J_2 = J_3$, or a $\text{Cliff}(7)$ -structure (but not both). As on every connected component M_α of M' , the number and the multiplicities of the eigenvalues of the operator $W_{X|X^\perp}$, $X \neq 0$, are constant, it follows from Remark 1 that the only case when M_α may potentially contain the points of the both kinds is when one of the eigenvalues of $W_{X|X^\perp}$, $X \neq 0$, on M_α has multiplicity 4 and the Clifford structure at every point $x \in M_\alpha$ is either $\text{Cliff}(3; J_1, J_2, J_3; \lambda_0, \eta_1, \eta_2, \eta_3)$ with $J_1 J_2 = J_3$, or $\text{Cliff}(7; J_1, \dots, J_7; \lambda_0 - 3\xi, \eta_1 + \xi, \eta_2 + \xi, \eta_3 + \xi, \xi, \xi, \xi)$, where $\eta_1, \eta_2, \eta_3 \neq 0$ (some of them can be equal) and $\xi \neq -\eta_i, 0$. The eigenvalues of $W_{X|X^\perp}$, $\|X\| = 1$, at every point $x \in M_\alpha$ are λ_0 , of multiplicity 4, and $\lambda_0 + 3\eta_i$. Let X be an arbitrary nowhere vanishing smooth vector field on a neighbourhood $\mathcal{U} \subset M_\alpha$ of a point $x \in M_\alpha$. Then the four-dimensional eigendistribution of the operator $W_{X|X^\perp}$ corresponding to the eigenvalue of multiplicity 4 is smooth, therefore its orthogonal complement, the distribution $\text{Span}(J_1 X, J_2 X, J_3 X)$ is also smooth. By assertion 3 of Lemma 3, there exist smooth anticommuting almost Hermitian structures $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$ on (a possibly smaller) neighbourhood \mathcal{U} such that $\text{Span}(\tilde{J}_1 X, \tilde{J}_2 X, \tilde{J}_3 X) = \text{Span}(J_1 X, J_2 X, J_3 X)$. By assertion 1 of Lemma 3, every \tilde{J}_i is a linear combination of the J_j 's: $\tilde{J}_i = \sum_{j=1}^3 a_{ij} J_j$, and moreover, the matrix (a_{ij}) must be orthogonal, as the \tilde{J}_i 's are anticommuting almost Hermitian structures. It follows that $\tilde{J}_1 \tilde{J}_2 \tilde{J}_3 = \pm J_1 J_2 J_3$. The operator on the left-hand side is smooth on \mathcal{U} , the one on the right-hand side is $\pm \text{id}_{\mathbb{R}^8}$, at the points where the Clifford structure is $\text{Cliff}(3)$ with $J_1 J_2 = J_3$, and is symmetric with trace zero, at the points where the Clifford structure is $\text{Cliff}(7)$ (which follows from the identity $J_4(J_1 J_2 J_3)J_4 = J_1 J_2 J_3$). Therefore all the point of \mathcal{U} either have a $\text{Cliff}(3)$ -structure with $J_1 J_2 = J_3$, or a $\text{Cliff}(7)$ -structure. In the both cases, the Clifford structure for W can be taken smooth: in the first case, we follow the arguments as in the first part of the proof, as $\nu < n - 1 - \nu$; in the second one, we apply assertion 3 of Lemma 3 to every eigendistribution of $W_{X|X^\perp}$.

Thus for any $x \in M'$, the Weyl tensor on a neighbourhood $\mathcal{U} = \mathcal{U}(x)$ has the form (1), with a constant ν and smooth λ_0, η_i and J_i . Then the curvature tensor has the form (7), with the operator ρ given by $\rho = \frac{1}{n-2} \text{Ric} + (\frac{1}{2} \lambda_0 - \frac{\text{scal}}{2(n-1)(n-2)}) \text{id}$, where Ric is the Ricci operator and scal is the scalar curvature. As λ_0 is a smooth function, the operator field ρ is also smooth. \square

Remark 2. In effect, the proof shows that if an algebraic curvature tensor \mathcal{R} field has a Clifford structure at every point of a Riemannian manifold, (and $\nu \neq 8$ when $n = 16$) then it has a Clifford structure of the same class of differentiability as \mathcal{R} on a neighbourhood of every generic point of the manifold.

Remark 3. As it follows from assertion 1 of Lemma 2 (in fact, from equation (4)), in the case $n = 8$, $\nu = 7$ we can replace in (7) ρ by $\rho - \frac{3}{2} f \text{id}$ and η_i by $\eta_i + f$, without changing R , where f is an arbitrary smooth function on \mathcal{U} (if we want the resulting Clifford structure to be $\text{Cliff}(7)$, we additionally require that $\eta_i + f$ is nowhere zero).

Let $x \in M'$ and let $\mathcal{U} = \mathcal{U}(x)$ be the neighbourhood of x defined in Lemma 4. By the second Bianchi identity, $(\nabla_U R)(X, Y)Y + (\nabla_Y R)(U, X)Y + (\nabla_X R)(Y, U)Y = 0$. Substituting R from (7) and using the fact that the operators J_i 's and their covariant derivatives are skew-symmetric and the operator ρ and

its covariant derivatives are symmetric we get:

$$\begin{aligned}
& \langle X, Y \rangle ((\nabla_U \rho)Y - (\nabla_Y \rho)U) + \|Y\|^2 ((\nabla_X \rho)U - (\nabla_U \rho)X) + \langle U, Y \rangle ((\nabla_Y \rho)X - (\nabla_X \rho)Y) \\
& + \langle (\nabla_Y \rho)U - (\nabla_U \rho)Y, Y \rangle X + \langle (\nabla_X \rho)Y - (\nabla_Y \rho)X, Y \rangle U + \langle (\nabla_U \rho)X - (\nabla_X \rho)U, Y \rangle Y \\
& + \sum_{i=1}^{\nu} 3(X(\eta_i) \langle J_i Y, U \rangle - U(\eta_i) \langle J_i Y, X \rangle) J_i Y \\
(8) \quad & + \sum_{i=1}^{\nu} Y(\eta_i) (2 \langle J_i U, X \rangle J_i Y + \langle J_i Y, X \rangle J_i U - \langle J_i Y, U \rangle J_i X) \\
& + \sum_{i=1}^{\nu} \eta_i ((3 \langle (\nabla_U J_i)X, Y \rangle + 3 \langle (\nabla_X J_i)Y, U \rangle + 2 \langle (\nabla_Y J_i)U, X \rangle) J_i Y \\
& + 3 \langle J_i X, Y \rangle (\nabla_U J_i)Y + 3 \langle J_i Y, U \rangle (\nabla_X J_i)Y + 2 \langle J_i U, X \rangle (\nabla_Y J_i)Y \\
& + \langle (\nabla_Y J_i)Y, X \rangle J_i U + \langle J_i Y, X \rangle (\nabla_Y J_i)U - \langle (\nabla_Y J_i)Y, U \rangle J_i X - \langle J_i Y, U \rangle (\nabla_Y J_i)X) = 0.
\end{aligned}$$

Taking the inner product of (8) with X and assuming X, Y and U to be orthogonal we obtain

$$\begin{aligned}
& \|X\|^2 \langle Q(Y), U \rangle + \|Y\|^2 \langle Q(X), U \rangle \\
(9) \quad & + \sum_{i=1}^{\nu} 3(X(\eta_i) \langle J_i Y, U \rangle - Y(\eta_i) \langle J_i X, U \rangle - U(\eta_i) \langle J_i Y, X \rangle) \langle J_i Y, X \rangle \\
& + \sum_{i=1}^{\nu} 3\eta_i ((2 \langle (\nabla_U J_i)X, Y \rangle + \langle (\nabla_X J_i)Y, U \rangle + \langle (\nabla_Y J_i)U, X \rangle) \langle J_i Y, X \rangle \\
& - \langle J_i Y, U \rangle \langle (\nabla_X J_i)X, Y \rangle - \langle J_i X, U \rangle \langle (\nabla_Y J_i)Y, X \rangle) = 0,
\end{aligned}$$

where $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the quadratic map defined by

$$(10) \quad \langle Q(X), U \rangle = \langle (\nabla_X \rho)U - (\nabla_U \rho)X, X \rangle.$$

Note that $\langle Q(X), X \rangle = 0$.

Lemma 5. *In the assumptions of Lemma 4, let $x \in M'$ and let \mathcal{U} be the corresponding neighbourhood of x . Suppose that if $n = 8$, then $\nu = 3$ and $J_1 J_2 = J_3$ on \mathcal{U} , and if $n = 16$, then $\nu \leq 4$. For every point $y \in \mathcal{U}$, identify $T_y M^n$ with the Euclidean space \mathbb{R}^n via a linear isometry. Then*

(i) *there exist vectors $m_i, b_{ij} \in \mathbb{R}^n$, $i, j = 1, \dots, \nu$, such that for all $X, Y, U \in \mathbb{R}^n$, and all $i = 1, \dots, \nu$,*

$$(11a) \quad Q(Y) = 3 \sum_{k=1}^{\nu} \langle m_k, Y \rangle J_k Y,$$

$$(11b) \quad (\nabla_X J_i)X = \eta_i^{-1} (\|X\|^2 m_i - \langle m_i, X \rangle X) + \sum_{j=1}^{\nu} \langle b_{ij}, X \rangle J_j X,$$

$$(11c) \quad b_{ij} + b_{ji} = \eta_i^{-1} J_j m_i + \eta_j^{-1} J_i m_j,$$

$$(11d) \quad \nabla \eta_i = 2 J_i m_i,$$

$$(11e) \quad \sum_{j \neq i} (\langle \eta_i b_{ij} + \eta_j b_{ji}, J_i Y \rangle J_j Y + \langle \eta_i b_{ij} + \eta_j b_{ji}, Y \rangle J_i J_j Y) = 0.$$

(ii) *the following equations hold:*

$$(12a) \quad (\nabla_Y \rho)U - (\nabla_U \rho)Y = \sum_{i=1}^{\nu} (2 \langle J_i Y, U \rangle m_i - \langle m_i, Y \rangle J_i U + \langle m_i, U \rangle J_i Y),$$

$$(12b) \quad b_{ij}(3 - \eta_i \eta_j^{-1}) + b_{ji}(3 - \eta_j \eta_i^{-1}) = 0, \quad i \neq j,$$

$$(12c) \quad J_i m_i = \eta_i p, \quad i = 1, \dots, \nu, \quad \text{for some } p \in \mathbb{R}^n.$$

Proof. (i) We split the proof of this assertions into the two cases: the *exceptional case*, when either $n = 6$, $\nu = 1$, or $n = 12$, $\nu = 3$, $J_1 J_2 = \pm J_3$, or $n = 8$, $\nu = 3$, $J_1 J_2 = J_3$, and the *generic case*: all the other Clifford structures considered in the lemma.

Generic case. From (9) we obtain

$$(13) \quad \|X\|^{-2} \langle Q(X), U \rangle + \|Y\|^{-2} \langle Q(Y), U \rangle = 0, \quad \text{for all } X \perp \mathcal{I}Y, X, Y \perp \mathcal{I}U, X, Y, U \neq 0.$$

We want to show that $\langle Q(X), U \rangle = 0$, for all $X \perp \mathcal{I}U$. This is immediate when $n > 3\nu + 3$. Indeed, for any $U \neq 0$ and any unit $X \perp \mathcal{I}U$, $\text{codim}(\mathcal{I}U + \mathcal{I}X) > \nu + 1$, so we can choose unit vectors

$Y_1, Y_2 \perp \mathcal{I}U + \mathcal{I}X$ such that $Y_1 \perp \mathcal{I}Y_2$. Then (13) implies that $\langle Q(X), U \rangle = -\langle Q(Y_1), U \rangle = \langle Q(Y_2), U \rangle = -\langle Q(X), U \rangle$.

Consider the case $n \leq 3\nu + 3$. By assertion (i) of Lemma 1, this could only happen when $n = 12$, $\nu = 3$ or $n = 24$, $\nu = 7$ (for the pairs (n, ν) belonging to the generic case), and in the both cases $n = 3\nu + 3$. Choose and fix an arbitrary $U \neq 0$ and consider the quadratic form $q(X) = \langle Q(X), U \rangle$ defined on the $(2\nu + 2)$ -dimensional space $L = (\mathcal{I}U)^\perp$. Assume $q \neq 0$. By (13), the restriction of q to the unit sphere of L is not a constant, so it attains its maximum (respectively minimum) on a great sphere S_1 (respectively S_2). The subspaces L_1 and L_2 defined by S_1 and S_2 are orthogonal. Moreover, by (13), $L_2 \supset (\mathcal{I}X)^\perp \cap L$, for any nonzero $X \in L_1$, which implies that $\dim L_2 \geq \nu + 1$. Similarly $\dim L_1 \geq \nu + 1$, so, as $L_1 \perp L_2$, $\dim L_1 = \dim L_2 = \nu + 1$, and $L = L_1 \oplus L_2$. It follows that for some $c > 0$, $q(X) = c(\|\pi_1 X\|^2 - \|\pi_2 X\|^2)$, where $\pi_i : L \rightarrow L_i$ is the orthogonal projection. Moreover, $L_2 = (\mathcal{I}X)^\perp \cap L$, for all nonzero $X \in L_1$, which means that the subspace $L_1 = L_2^\perp \cap L$ (and similarly L_2) is $\pi\mathcal{I}$ -invariant, where $\pi : \mathbb{R}^n \rightarrow L$ is the orthogonal projection, and even more: $\pi\mathcal{I}X = L_\alpha$, for every nonzero $X \in L_\alpha$, $\alpha = 1, 2$, by the dimension count. Let $X = X_1 + X_2$, $Y = Y_1 + Y_2 \in L$, where $X_\alpha = \pi_\alpha X$, $Y_\alpha = \pi_\alpha Y$. The condition $Y \perp \mathcal{I}X$ is equivalent to $\langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle = \langle \pi J_i X_1, Y_1 \rangle + \langle \pi J_i X_2, Y_2 \rangle = 0$, for all $i = 1, \dots, \nu$. Take arbitrary orthonormal bases for L_1 and for L_2 and denote $M_\alpha(X_\alpha)$, $\alpha = 1, 2$, the $(\nu + 1) \times (\nu + 1)$ -matrix whose columns relative to the chosen basis for L_α are $X_\alpha, \pi J_1 X_\alpha, \dots, \pi J_\nu X_\alpha$. Then $Y \perp \mathcal{I}X$ if and only if $M_1(X_1)^t Y_1 = -M_2(X_2)^t Y_2$. Since for $\alpha = 1, 2$, and any nonzero $X_\alpha \in L_\alpha$, the columns of $M_\alpha(X_\alpha)$ span L_α , we obtain $Y_2 = -(M_2(X_2)^t)^{-1} M_1(X_1)^t Y_1$, for any $X_2 \neq 0$. Then, as $q(X) = c(\|X_1\|^2 - \|X_2\|^2)$, $q(Y) = c(\|Y_1\|^2 - \|Y_2\|^2)$, equation (13) implies $\|Y_1\|^2 \|X_1\|^2 - \|Y_2\|^2 \|X_2\|^2 = 0$, so $\|Y_1\|^2 \|X_1\|^2 - \|(M_2(X_2)^t)^{-1} M_1(X_1)^t Y_1\|^2 \|X_2\|^2 = 0$, for any $X_1, Y_1 \in L_1$ and any nonzero $X_2 \in L_2$. It follows that $\|X_1\|^2 (M_1(X_1)^t M_1(X_1))^{-1} = \|X_2\|^2 (M_2(X_2)^t M_2(X_2))^{-1}$, for any nonzero $X_\alpha \in L_\alpha$. Thus for some positive definite symmetric $(\nu + 1) \times (\nu + 1)$ -matrix T , we have $M_\alpha(X_\alpha)^t M_\alpha(X_\alpha) = \|X_\alpha\|^2 T$, for all $X_\alpha \in L_\alpha$, $\alpha = 1, 2$. Then for any $X = X_1 + X_2 \in L$, $X_\alpha \in L_\alpha$, and any $i = 1, \dots, \nu$, $\|\pi J_i X\|^2 = \|\pi J_i X_1\|^2 + \|\pi J_i X_2\|^2 = (M_1(X_1)^t M_1(X_1) + M_2(X_2)^t M_2(X_2))_{ii} = T_{ii}(\|X_1\|^2 + \|X_2\|^2) = T_{ii} \|X\|^2$. On the other hand, for any $X \in L$, $\pi J_i X = J_i X - \|U\|^{-2} \sum_{j=1}^\nu \langle J_i X, J_j U \rangle J_j U$, so $\|\pi J_i X\|^2 = \|X\|^2 - \|U\|^{-2} \sum_{j=1}^\nu \langle J_i X, J_j U \rangle^2$. It follows that $\|X\|^2 \|U\|^2 (1 - T_{ii}) = \sum_{j=1}^\nu \langle J_i X, J_j U \rangle^2 = \sum_{j=1}^\nu \langle X, J_i J_j U \rangle^2$, for an arbitrary $X \in L$. As $\dim L = 2\nu + 2 > \nu$, we can choose a nonzero $X \in L$ orthogonal to the ν vectors $J_i J_j U$, $j = 1, \dots, \nu$. This implies $T_{ii} = 1$, so $X \perp J_i J_j U$, for all $i, j = 1, \dots, \nu$ and all $X \in L = (\mathcal{I}U)^\perp$. Therefore $J_i J_j U \in \mathcal{I}U$, for all $i, j = 1, \dots, \nu$ and all $U \in \mathbb{R}^n$ for which the quadratic form $q(X) = \langle Q(X), U \rangle$ defined on $(\mathcal{I}U)^\perp$ is nonzero. If this is true for at least one U , then this is true for a dense subset of \mathbb{R}^n , which implies that $J_i J_j U \in \mathcal{I}U$, for all $i, j = 1, \dots, \nu$ and all $U \in \mathbb{R}^n$. Then by assertion 1 of Lemma 3, for $i \neq j$, $J_i J_j U = \sum_{k=1}^\nu a_{ijk} J_k U$, for some constants a_{ijk} , which implies that $\langle J_k J_i J_j U, U \rangle = a_{ijk} \|U\|^2$, so for all the triples of pairwise distinct i, j, k , the symmetric operator $J_k J_i J_j$ on \mathbb{R}^n is a multiple of the identity. This is impossible when $\nu > 3$ (as for $l \neq i, j, k$, the operator $J_l J_k J_i J_j$ must be orthogonal and symmetric). The only remaining cases are $n = 12$, $\nu = 3$, with $J_1 J_2 J_3 = \pm \text{id}$, and $n = 6$, $\nu = 1$, which are considered under the exceptional case below.

Thus $\langle Q(X), U \rangle = 0$, for $X \perp \mathcal{I}U$, so $Q(X) \in \mathcal{I}X$, for all $X \in \mathbb{R}^n$. By assertion 1 of Lemma 3 (and the fact that $\langle Q(X), X \rangle = 0$), this implies equation (11a), with some vectors $m_i \in \mathbb{R}^n$.

To prove (11b) and (11c), we first show that for an arbitrary $X \neq 0$, there is a dense subset of the Y 's in $(\mathcal{I}X)^\perp$ such that $\mathcal{J}X \cap \mathcal{J}Y = 0$. This follows from the dimension count (compare to [N1, Lemma 3.2 (1)]). For $X \neq 0$, define the cone $\mathcal{C}X = \{J_u J_v X : u, v \in \mathbb{R}^\nu\}$ (see (2)). As $\dim \mathcal{C}X \leq 2\nu - 1 < n - (\nu + 1) = \dim(\mathcal{I}X)^\perp$ (the inequality in the middle follows from assertion (i) of Lemma 1), the complement to $\mathcal{C}X$ is dense in $(\mathcal{I}X)^\perp$. This complement is the required subset, as the condition $Y \notin \mathcal{C}X$ is equivalent to $\mathcal{J}X \cap \mathcal{J}Y = 0$. Substituting such X, Y into (9) we obtain by (11a):

$$\sum_{i=1}^\nu (\|X\|^2 \langle m_i, Y \rangle - \eta_i \langle (\nabla_X J_i) X, Y \rangle) J_i Y + \sum_{i=1}^\nu (\|Y\|^2 \langle m_i, X \rangle - \eta_i \langle (\nabla_Y J_i) Y, X \rangle) J_i X = 0.$$

As $\mathcal{J}X \cap \mathcal{J}Y = 0$, all the coefficients vanish, so $\|X\|^2 \langle m_i, Y \rangle - \eta_i \langle (\nabla_X J_i) X, Y \rangle = 0$, for all $X \in \mathbb{R}^n$, all $i = 1, \dots, \nu$, and all Y from a dense subset of $(\mathcal{I}X)^\perp$, which implies that $(\nabla_X J_i) X - \eta_i^{-1} \|X\|^2 m_i \in \mathcal{I}X$, for all $X \in \mathbb{R}^n$. Equation (11b) then follows from assertion 1 of Lemma 3. Equation (11c) follows from (11b) and the fact that $\langle (\nabla_X J_i) X, J_j X \rangle + \langle (\nabla_X J_j) X, J_i X \rangle = 0$.

To prove (11d) and (11e), substitute $X = J_k Y$, $U \perp X, Y$ into (9). Consider the first term in the second summation. As $\langle J_i Y, X \rangle = \|Y\|^2 \delta_{ik}$, that term equals $3\eta_k(2\langle(\nabla_U J_k)X, Y\rangle + \langle(\nabla_X J_k)Y, U\rangle + \langle(\nabla_Y J_k)U, X\rangle)\|Y\|^2$. As J_k is orthogonal and skew-symmetric, $\langle(\nabla_U J_k)X, Y\rangle = \langle(\nabla_U J_k)J_k Y, Y\rangle = -\langle J_k(\nabla_U J_k)Y, Y\rangle = \langle(\nabla_U J_k)Y, J_k Y\rangle = 0$. Next, $\langle(\nabla_Y J_k)U, X\rangle = -\langle(\nabla_Y J_k)J_k Y, U\rangle = \langle J_k(\nabla_Y J_k)Y, U\rangle = \langle(\eta_k^{-1}\|Y\|^2 J_k m_k + \sum_{j=1}^{\nu} \langle b_{kj}, Y\rangle J_k J_j Y, U\rangle$ by (11b). Similarly, as $Y = -J_k X$, it follows from (11b) that $\langle(\nabla_X J_k)Y, U\rangle = \langle J_k(\nabla_X J_k)X, U\rangle = \langle J_k(\eta_k^{-1}(\|X\|^2 m_k - \langle m_k, X\rangle X) + \sum_{j=1}^{\nu} \langle b_{kj}, X\rangle J_j X), U\rangle = \langle\eta_k^{-1}\|Y\|^2 J_k m_k + \sum_{j \neq k} \langle b_{kj}, J_k Y\rangle J_j Y - \langle b_{kk}, J_k Y\rangle J_k Y, U\rangle$. Substituting this into (9) and using (11a) and (11b) we obtain after simplification:

$$(14) \quad \|Y\|^2 \langle 2J_k m_k - U(\eta_k) \rangle + \sum_{j=1}^{\nu} \langle \eta_k b_{kj} + \eta_j b_{jk}, \langle J_j Y, U \rangle J_k Y + \langle J_k J_j Y, U \rangle Y \rangle = 0.$$

By [N1, Lemma 3.2(3)], for all $U \in \mathbb{R}^n$, we can find a nonzero Y such that $U \perp \mathcal{J}Y + \mathcal{J}J_k Y$. Substituting such a Y into (14) proves (11d). Then (14) simplifies to (11e).

Exceptional case (either $n = 6$, $\nu = 1$, or $n = 12$, $\nu = 3$, $J_1 J_2 = \pm J_3$, or $n = 8$, $\nu = 3$, $J_1 J_2 = J_3$).

In all these cases, the Clifford structure has the following “ J^2 -property”: for every $X \in \mathbb{R}^n$, $\mathcal{I}IX = \mathcal{J}IX = IX$. In particular, if $Y \perp IX$, then $\mathcal{I}Y \perp IX$.

Substitute $X = J_k U$ and $Y \perp IX = \mathcal{I}U$ to (8) and take the inner product of the resulting equation with $J_k Y$. Using the fact that $\langle(\nabla_Y J_k)U, J_k U\rangle = \langle(\nabla_Y J_k)Y, J_k Y\rangle = 0$ and the J^2 -property we get

$$-J_k((\nabla_{J_k U} \rho)U - (\nabla_U \rho)J_k U) + 2\|U\|^2 \nabla \eta_k + 3\eta_k((\nabla_U J_k)J_k U - (\nabla_{J_k U} J_k)U) \in \mathcal{I}U.$$

The expression $F(U)$ on the left-hand side is a quadratic map from \mathbb{R}^n to itself. By assertion 1 of Lemma 3, $F(U)$ is a linear combination of $U, J_1 U, \dots, J_{\nu} U$ whose coefficients are linear forms of U . In particular, the cubic polynomial $\langle F(U), J_k U \rangle$ must be divisible by $\|U\|^2$. As J_k is orthogonal and skew-symmetric, $\langle(\nabla_U J_k)J_k U - (\nabla_{J_k U} J_k)U, J_k U\rangle = 0$, so there exists a vector $m_k \in \mathbb{R}^n$ such that $\langle(\nabla_{J_k U} \rho)U - (\nabla_U \rho)J_k U, U\rangle = -3\|U\|^2 \langle m_k, U \rangle$. It follows that the quadratic map Q defined by (10) satisfies $\langle Q(U), J_k U \rangle = 3\|U\|^2 \langle m_k, U \rangle$, for all $U \in \mathbb{R}^8$ and all $k = 1, \dots, \nu$. As $\langle Q(U), U \rangle = 0$, we can define a quadratic map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for all $U \in \mathbb{R}^n$,

$$(15) \quad Q(U) = T(U) + 3 \sum_{k=1}^{\nu} \langle m_k, U \rangle J_k U, \quad T(U) \perp \mathcal{I}U.$$

Taking $U = J_k X$, $X, U \perp \mathcal{I}Y$ in (9) and using (15) we obtain $-J_k T(Y) + 3\|Y\|^2 m_k - 3\eta_k(\nabla_Y J_k)Y \in \mathcal{I}Y$. From assertion 1 of Lemma 3 it follows that the expression on the left-hand side is a linear combination of $Y, J_1 Y, \dots, J_{\nu} Y$ whose coefficients are linear forms of Y , so for some vectors $b_{ij} \in \mathbb{R}^n$,

$$(16) \quad (\nabla_Y J_i)Y = \eta_i^{-1}(m_i \|Y\|^2 - \langle m_i, Y \rangle Y) - (3\eta_i)^{-1} J_i T(Y) + \sum_{j=1}^{\nu} \langle b_{ij}, Y \rangle J_j Y.$$

As $\langle(\nabla_Y J_i)Y, J_j Y\rangle$ is antisymmetric in i and j and $J_i T(Y) \perp \mathcal{I}Y$ by (15) and the J^2 -property, the b_{ij} 's satisfy (11c).

Take $X = J_k Y$, $U \perp \mathcal{I}Y = \mathcal{I}X$ in (9). As $\langle(\nabla_U J_k)J_k Y, Y\rangle = 0$, $\langle(\nabla_Y J_k)U, X\rangle = -\langle(\nabla_Y J_k)J_k Y, U\rangle = \langle J_k(\nabla_Y J_k)Y, U\rangle$, and similarly, $\langle(\nabla_X J_k)Y, U\rangle = -\langle(\nabla_X J_k)J_k X, U\rangle = \langle J_k(\nabla_X J_k)X, U\rangle$, we obtain from (15, 16) after simplification that

$$(17) \quad 2T(Y) + 2T(J_k Y) - 3\|Y\|^2(\nabla \eta_k - 2J_k m_k) \in \mathcal{I}Y.$$

In the case $n = 6$, $\nu = 1$, we can prove the remaining identities (11a, 11b, 11d, 11e) of assertion (i) as follows. Taking in (9) nonzero X, Y, U such that the subspaces $\mathcal{I}X, \mathcal{I}Y$ and $\mathcal{I}U$ are mutually orthogonal we obtain by (15) $\|X\|^{-2} \langle T(X), U \rangle + \|Y\|^{-2} \langle T(Y), U \rangle = 0$ (which is, essentially, (13)). Replacing Y by $J_1 Y$ and using (17) we get $2T(X) + 3\|X\|^2(\nabla \eta_1 - 2J_1 m_1) \in \mathcal{I}X$. The same is true with X replaced by $J_1 X$. Then by (17), $\nabla \eta_1 - 2J_1 m_1 \in \mathcal{I}X$, for all $X \in \mathbb{R}^6$, so $\nabla \eta_1 - 2J_1 m_1 = 0$ (which is (11d)). Then $T(X) \in \mathcal{I}X$, hence $T(X) = 0$, as $T(X) \perp \mathcal{I}X$ by (15). Now (11a) follows from (15), (11b) follows from (16), and (11e) is trivially satisfied, as $\nu = 1$.

In the cases $n = 8, 12$, $\nu = 3$, $J_1 J_2 = J_3$ (if $J_1 J_2 = -J_3$, we replace J_3 by $-J_3$, without changing the curvature tensor (7)), we argue as follows. Adding equation (17) with $k = 1$ and with $k = 2$ and then subtracting (17) with $k = 3$ and Y replaced by $J_1 Y$ we get $4T(Y) - 3\|Y\|^2((\nabla \eta_1 - 2J_1 m_1) + (\nabla \eta_2 - 2J_2 m_2) - (\nabla \eta_3 - 2J_3 m_3)) \in \mathcal{I}Y$. This remains true under a cyclic permutation of the indices 1, 2, 3, which implies $(\nabla \eta_k - 2J_k m_k) - (\nabla \eta_i - 2J_i m_i) \in \mathcal{I}Y$, for all $i, k = 1, 2, 3$ and all $Y \in \mathbb{R}^n$. Then

$\nabla\eta_k - 2J_k m_k = \nabla\eta_i - 2J_i m_i = \frac{4}{3}V$, for some vector $V \in \mathbb{R}^n$, and $T(Y) - \|Y\|^2 V \in \mathcal{I}Y$ from the above. As $T(Y) \perp \mathcal{I}Y$ by (15), we obtain $T(Y) = \|Y\|^2 V - \langle Y, V \rangle Y - \sum_{i=1}^3 \langle J_i Y, V \rangle J_i Y$, so

$$(18) \quad \begin{aligned} \nabla\eta_i &= 2J_i m_i + \frac{4}{3}V, & Q(Y) &= \|Y\|^2 V - \langle Y, V \rangle Y + \sum_{j=1}^3 \langle 3m_j + J_j V, Y \rangle J_j Y, \\ (\nabla_Y J_i)Y &= (3\eta_i)^{-1}(\|Y\|^2(3m_i - J_i V) - \langle 3m_i - J_i V, Y \rangle Y + \sum_{j=1}^3 \langle 3\eta_i b_{ij} - J_j J_i V, Y \rangle J_j Y) \end{aligned}$$

(the second equation follows from (15); the third one, from (16) and the fact that $J_1 J_2 = J_3$).

Substitute $X = J_k Y$ into (9) again, with an arbitrary $U \perp X, Y$. Using (18) and the fact that the J_i 's are skew-symmetric, orthogonal and anticommute, we obtain after simplification:

$$\sum_{i=1}^3 \langle 3a_{ik} - 2J_i J_k V, J_k Y \rangle J_i Y + \sum_{i=1}^3 \langle 3a_{ik} - 2J_i J_k V, Y \rangle J_k J_i Y \in \text{Span}(Y, J_k Y),$$

where $a_{ik} = \eta_k b_{ki} + \eta_i b_{ik}$. Taking $k = 1$ and using the fact that $J_1 J_2 = J_3$ we get from the coefficient of $J_2 Y$: $3J_1 a_{12} - 4J_2 V + 3a_{13} = 0$, so $4V = -3J_2 a_{13} + 3J_3 a_{12}$. Cyclically permuting the indices 1, 2, 3 and using the fact that $a_{ik} = a_{ki}$ we obtain $V = 0$, which implies (11e). As $V = 0$, equations (11a, 11d, 11b) follow from (18).

(ii) By (10) and (11a), $\langle (\nabla_X \rho)U - (\nabla_U \rho)X, X \rangle = 3 \sum_{i=1}^\nu \langle m_i, X \rangle \langle J_i X, U \rangle$, for all $X, U \in \mathbb{R}^n$. Polarizing this equation and using the fact that the covariant derivative of ρ is symmetric we obtain $\langle (\nabla_X \rho)U, Y \rangle + \langle (\nabla_Y \rho)U, X \rangle - 2\langle (\nabla_U \rho)Y, X \rangle = 3 \sum_{i=1}^\nu (\langle m_i, Y \rangle \langle J_i X, U \rangle + \langle m_i, X \rangle \langle J_i Y, U \rangle)$. Subtracting the same equation, with Y and U interchanged, we get $\langle (\nabla_Y \rho)U - (\nabla_U \rho)Y, X \rangle = \sum_{i=1}^\nu (2\langle m_i, X \rangle \langle J_i Y, U \rangle + \langle m_i, Y \rangle \langle J_i X, U \rangle - \langle m_i, U \rangle \langle J_i X, Y \rangle)$, which proves (12a).

To establish (12b), substitute $X \perp \mathcal{I}Y$, $U = J_k Y$ into (8). Using the equations of assertion (i) and (12a) we obtain after simplification:

$$3(\nabla_X J_k)Y - (\nabla_Y J_k)X = -3\eta_k^{-1} \langle m_k, Y \rangle X + \sum_{i=1}^\nu \eta_k^{-1} \langle \eta_i b_{ik} + 2\delta_{ik} J_k m_k, Y \rangle J_i X \quad \text{mod } (\mathcal{I}Y).$$

Subtracting three times polarized equation (11b) (with $i = k$) and solving for $(\nabla_Y J_k)X$ we get

$$(19) \quad (\nabla_Y J_k)X = \sum_{i=1}^\nu \frac{1}{4} \eta_k^{-1} \langle 3\eta_k b_{ki} - \eta_i b_{ik} - 2\delta_{ik} J_k m_k, Y \rangle J_i X \quad \text{mod } (\mathcal{I}Y),$$

for all $X \perp \mathcal{I}Y$. Choose $s \neq k$ and define the subset $S_{ks} \subset \mathbb{R}^n \oplus \mathbb{R}^n$ by $S_{ks} = \{(X, Y) : X, Y \neq 0, X, J_k X, J_s X \perp \mathcal{I}Y\}$. It is easy to see that $(X, Y) \in S_{ks} \Leftrightarrow (Y, X) \in S_{ks}$ and that replacing $\mathcal{I}Y$ by $\mathcal{I}Y$ in the definition of S_{ks} gives the same set S_{ks} . Moreover, the set $\{X : (X, Y) \in S_{ks}\}$ (and hence the set $\{Y : (X, Y) \in S_{ks}\}$) spans \mathbb{R}^n . If $n = 8$, $\nu = 3$, $J_1 J_2 = J_3$, this easily follows from the J^2 -property; in all the other cases, from [N1, Lemma 3.2 (4)]. For $(X, Y) \in S_{ks}$, take the inner product of (19) with $J_s X$. Since $\langle (\nabla_Y J_k)X, J_s X \rangle$ is antisymmetric in k and s , we get $\langle (3 - \eta_k \eta_s^{-1})b_{ks} + (3 - \eta_s \eta_k^{-1})b_{sk}, Y \rangle = 0$, for a set of the Y 's spanning \mathbb{R}^n . This proves (12b).

To prove (12c), we apply assertion 2 of Lemma 3 to equation (11e). If $\nu = 1$, there is nothing to prove (in fact, if $\nu = 1$ and $n \geq 8$, the claim of Theorem 3 follows from [BG1, Theorem 1.1]). If $\eta_i b_{ij} + \eta_j b_{ji} = 0$ for all $i \neq j$, then by (12b), $b_{ij} + b_{ji} = 0$ for all $i \neq j$, so by (11c), $\eta_i^{-1} J_j m_i = -\eta_j^{-1} J_i m_j$. Acting by $J_i J_j$ we obtain that the vector $\eta_i^{-1} J_i m_i$ is the same, for all $i = 1, \dots, \nu$.

The only remaining possibility is $\nu = 3$, $J_1 J_2 = J_3$ (if $J_1 J_2 = -J_3$ we can replace J_3 by $-J_3$ without changing the curvature tensor (7)), and $\eta_k b_{ki} + \eta_i b_{ik} = J_j v$, for all the triples $\{i, j, k\} = \{1, 2, 3\}$, where $v \neq 0$. We will show that this leads to a contradiction. Note that by (3), the existence of a Cliff(3)-structure implies that n is divisible by 4, so by the assumption of the lemma, $n \geq 8$.

If $\eta_i = \eta_k$ for some $i \neq k$, then from (12b) and $\eta_k b_{ki} + \eta_i b_{ik} = J_j v$ it follows that $v = 0$, a contradiction. Otherwise, if the η_i 's are pairwise distinct, we get $b_{ik} = (3\eta_i - \eta_k)(4\eta_i(\eta_i - \eta_k))^{-1} J_j v$ for $\{i, j, k\} = \{1, 2, 3\}$. Substituting this to (11c) and acting by J_j on the both sides we get $\eta_i^{-1} J_i m_i - \eta_k^{-1} J_k m_k = \frac{1}{4} \varepsilon_{ik} (\eta_i^{-1} + \eta_k^{-1}) v$, for $\{i, j, k\} = \{1, 2, 3\}$, where for $i \neq k$ we define $\varepsilon_{ik} = \pm 1$ by $J_i J_k = \varepsilon_{ik} J_j$. It is easy to see that $\varepsilon_{jk} = -\varepsilon_{kj}$ and $\varepsilon_{jk} = \varepsilon_{ij}$, where $\{i, j, k\} = \{1, 2, 3\}$. Then $\sum_{i=1}^3 \eta_i^{-1} = 0$ and $\eta_i^{-1} J_i m_i = \frac{1}{12} \varepsilon_{jk} (\eta_j^{-1} - \eta_k^{-1}) v + w$, for some $w \in \mathbb{R}^n$. It then follows from (11d) that $\nabla\eta_i = \frac{1}{6} \varepsilon_{jk} \eta_i (\eta_j^{-1} - \eta_k^{-1}) v + 2\eta_i w$, which implies $\nabla \ln |\eta_1 \eta_2 \eta_3| = 6w$ and $\nabla \ln |\eta_i \eta_j^{-1}| = -\frac{1}{2} \varepsilon_{ij} \eta_k^{-1} v$. Let $\mathcal{U}' \subset \mathcal{U}$ be a neighbourhood of x on which $\nabla \ln |\eta_1 \eta_2^{-1}| \neq 0$. Then v is a nowhere vanishing smooth vector field on \mathcal{U}' . Multiplying

the metric on \mathcal{U} by a function e^f changes neither the Weil tensor, nor the J_i 's, and multiplies every η_i by e^{-f} and ∇ acting on functions by e^{-f} . Taking $f = \frac{1}{3} \ln |\eta_1 \eta_2 \eta_3|$ we can assume that $w = 0$ on \mathcal{U}' , so that $C = \eta_1 \eta_2 \eta_3$ is a constant. Then, as $\sum_{i=1}^3 \eta_i^{-1} = 0$, we get $\nabla \eta_i = \pm \frac{1}{6} \sqrt{1 - 4C^{-1} \eta_i^3} v$. It follows that $v = \nabla t$ for some smooth function $t : \mathcal{U}' \rightarrow \mathbb{R}$ such that $\eta_i = -36C \wp(t + c_i)$, where \wp is the Weierstrass function satisfying $(\frac{d}{dt} \wp(t))^2 = 4\wp(t)^3 + 6^{-6} C^{-2}$ and $c_i \in \mathbb{R}$. Summarizing the identities of this paragraph, we have pointwise pairwise nonequal functions $\eta_i : \mathcal{U}' \rightarrow \mathbb{R} \setminus \{0\}$ satisfying

$$(20) \quad \begin{aligned} v = \nabla t \neq 0, \quad \nabla \eta_i &= \frac{1}{6} \varepsilon_{jk} \eta_i (\eta_j^{-1} - \eta_k^{-1}) v, \quad \sum_{i=1}^3 \eta_i^{-1} = 0, \quad \prod_{i=1}^3 \eta_i = C = \text{const}, \\ m_i &= -\frac{1}{12} \varepsilon_{jk} \eta_i (\eta_j^{-1} - \eta_k^{-1}) J_i v, \quad b_{ii} = \frac{1}{12} \varepsilon_{jk} (\eta_j^{-1} - \eta_k^{-1}) v, \quad b_{ij} = (3\eta_i - \eta_j)(4\eta_i(\eta_i - \eta_j))^{-1} J_k v, \end{aligned}$$

for $\{i, j, k\} = \{1, 2, 3\}$, where we used (11c) to compute b_{ii} . Then equation (19) simplifies to $(\nabla_Y J_k)X = \sum_{i \neq k} \frac{1}{2} (\eta_k - \eta_i)^{-1} \langle J_j v, Y \rangle J_i X \pmod{\mathcal{I}Y}$, for all $X \perp \mathcal{I}Y$. By the J^2 -property, $\mathcal{I}Y \perp \mathcal{I}X$, so to find the “mod($\mathcal{I}Y$)”-part, we have to compute the inner products of $(\nabla_Y J_k)X$ with $Y, J_1 Y, J_2 Y, J_3 Y$. Since $\langle (\nabla_Y J_k)X, Y \rangle = -\langle (\nabla_Y J_k)Y, X \rangle$, $\langle (\nabla_Y J_k)X, J_k Y \rangle = -\langle (\nabla_Y J_k)J_k Y, X \rangle = \langle J_k (\nabla_Y J_k)Y, X \rangle$, and $\langle (\nabla_Y J_k)X, J_i Y \rangle = -\langle (\nabla_Y J_k)J_i Y, X \rangle = -\langle (\varepsilon_{ki} (\nabla_Y J_j) - J_k (\nabla_Y J_i))Y, X \rangle$ (from $J_k J_i = \varepsilon_{ki} J_j$), these products can be found using (11b). Simplifying by (20) we get

$$\begin{aligned} (\nabla_Y J_k)X &= \frac{1}{12} \varepsilon_{ij} (\eta_i^{-1} - \eta_j^{-1}) (\langle J_k v, X \rangle Y + \langle v, X \rangle J_k Y) \\ &\quad + \frac{1}{4} \eta_k^{-1} \sum_{i \neq k} \langle J_j v, X \rangle J_i Y + \sum_{i \neq k} \frac{1}{2} (\eta_k - \eta_i)^{-1} \langle J_j v, Y \rangle J_i X, \end{aligned}$$

for all $X \perp \mathcal{I}Y$, where $\{i, j, k\} = \{1, 2, 3\}$. To compute $(\nabla_Y J_k)X$ when $X \in \mathcal{I}Y$ we again use (11b) and the fact that $(\nabla_Y J_k)J_k = -J_k (\nabla_Y J_k)$ and $(\nabla_Y J_k)J_i = \varepsilon_{ki} (\nabla_Y J_j) - J_k (\nabla_Y J_i)$, for $\{i, j, k\} = \{1, 2, 3\}$. Simplifying by (20) and using the above equation we get after some calculations:

$$\begin{aligned} (\nabla_Y J_k)X &= \frac{1}{12} \varepsilon_{ij} (\eta_i^{-1} - \eta_j^{-1}) (\langle J_k v, X \rangle Y + \langle v, X \rangle J_k Y - \langle X, Y \rangle J_k v - \langle X, J_k Y \rangle v) \\ &\quad + \frac{1}{4} \eta_k^{-1} \sum_{i \neq k} (\langle J_j v, X \rangle J_i Y - \langle J_i Y, X \rangle J_j v) + \sum_{i \neq k} \frac{1}{2} (\eta_k - \eta_i)^{-1} \langle J_j v, Y \rangle J_i X, \end{aligned}$$

for all $X, Y \in \mathbb{R}^n$, where $\{i, j, k\} = \{1, 2, 3\}$. Let for $a, b \in \mathbb{R}^n$, $a \wedge b$ be the skew-symmetric operator defined by $(a \wedge b)X = \langle a, X \rangle b - \langle b, X \rangle a$. Then the above equation can be written in the form $\nabla_Y J_k = \frac{1}{12} \varepsilon_{ij} (\eta_i^{-1} - \eta_j^{-1}) (J_k v \wedge Y + v \wedge J_k Y) + \frac{1}{4} \eta_k^{-1} \sum_{i \neq k} J_j v \wedge J_i Y + \sum_{i \neq k} \frac{1}{2} (\eta_k - \eta_i)^{-1} \langle J_j v, Y \rangle J_i$, that is,

$$(21) \quad \begin{aligned} \nabla_Y J_k &= [J_k, AY], \quad AY = \frac{1}{2} \sum_{i=1}^3 \lambda_i J_i Y \wedge J_i v + \sum_{i=1}^3 \omega_i \langle J_i v, Y \rangle J_i, \\ \lambda_i &= \frac{1}{6} \varepsilon_{jk} (\eta_j^{-1} - \eta_k^{-1}), \quad \omega_i = \frac{1}{4} \varepsilon_{jk} (\eta_k - \eta_j)^{-1} \quad \text{for } \{i, j, k\} = \{1, 2, 3\}, \end{aligned}$$

where we used the fact that $[J_k, a \wedge b] = J_k a \wedge b + a \wedge J_k b$ and $[J_k, J_i] = 2\varepsilon_{ki} J_j$, for $\{i, j, k\} = \{1, 2, 3\}$. By the Ricci formula, $\nabla_{Z,Y}^2 J_k - \nabla_{Y,Z}^2 J_k = [J_k, R(Y, Z)]$, where the tensor field $\nabla^2 J_k$ is defined by $\nabla_{Z,Y}^2 J_k = \nabla_Z (\nabla_Y J_k) - \nabla_{\nabla_Z Y} J_k$ for vector fields Y, Z on \mathcal{U}' . As $\nabla_Y J_k = [J_k, AY]$ by (21), this is equivalent to the fact that the operator $F(Y, Z) = (\nabla_Z A)Y - (\nabla_Y A)Z - [AY, AZ] - R(Y, Z)$ commutes with all the J_s 's, for all $Y, Z \in \mathbb{R}^n$ and all $s = 1, 2, 3$. As by (7), $R(Y, Z) = Y \wedge \rho Z + \rho Y \wedge Z + \sum_{i=1}^3 \eta_i (J_i Y \wedge J_i Z + 2 \langle J_i Y, Z \rangle J_i)$, we obtain using (21) and the identities $[a \wedge b, c \wedge d] = \langle a, c \rangle d \wedge b - \langle a, c \rangle d \wedge b - \langle b, d \rangle c \wedge a + \langle b, d \rangle c \wedge a$, $[J_s, a \wedge b] = J_s a \wedge b + a \wedge J_s b$:

$$(22) \quad \begin{aligned} F(Y, Z) &= V(Y, Z) + \sum_{i=1}^3 \langle K_i Y, Z \rangle J_i + S(Y, Z), \quad \text{where } S(Y, Z) \in (\mathcal{I}Y + \mathcal{I}Z) \wedge \mathbb{R}^n \quad \text{and} \\ V(Y, Z) &= -\frac{1}{2} \sum_{i=1}^3 \langle J_i Z, Y \rangle (\lambda_i^2 v \wedge J_i v + \varepsilon_{jk} (\lambda_j \lambda_k - \lambda_i \lambda_k - \lambda_j \lambda_i) J_j v \wedge J_k v) \in \mathcal{I}v \wedge \mathcal{I}v, \end{aligned}$$

where for subspaces $L_1, L_2 \subset \mathbb{R}^n$, we denote $L_1 \wedge L_2$ the subspace of the space $\mathfrak{o}(n)$ of the skew-symmetric operators on \mathbb{R}^n defined by $L_1 \wedge L_2 = \text{Span}(a \wedge b : a \in L_1, b \in L_2)$. Note that if L_1, L_2 are \mathcal{J} -invariant (that is, $\mathcal{J}L_\alpha \subset L_\alpha$), then $L_1 \wedge L_2$ is ad \mathcal{J} -invariant, that is, $[J_s, L_1 \wedge L_2] \subset L_1 \wedge L_2$.

From (21) and using the fact that $\omega_i \lambda_i = (24C)^{-1} \eta_i$, $\frac{d}{dt} \omega_i = 4\omega_i^2 + (12C)^{-1} \eta_i$ and $\sum_i \omega_i^{-1} = 0$, which follow from (20, 21), we obtain

$$(23) \quad K_i = -\omega_i ((4\omega_i + \lambda_i) v \wedge J_i v + 4\varepsilon_{jk} (\omega_j + \omega_k) J_j v \wedge J_k v + \lambda_i (48C + \|v\|^2) J_i + (J_i H + H J_i)),$$

where $\{i, j, k\} = \{1, 2, 3\}$, and H is the symmetric operator associated to the Hessian of the function t (that is, $\langle HY, Z \rangle = Y(Zt) - (\nabla_Y)Zt$, for vector fields Y, Z on \mathcal{U}').

As $[F(Y, Z), J_s] = 0$ and the subspace $\mathcal{I}Y + \mathcal{I}Z$ is \mathcal{J} -invariant (hence $(\mathcal{I}Y + \mathcal{I}Z) \wedge \mathbb{R}^n$ is $\text{ad}_{\mathcal{J}}$ -invariant), it follows from (22) that for all $Y, Z \in \mathbb{R}^n$ and all $s = 1, 2, 3$,

$$(24) \quad [V(Y, Z), J_s] + \sum_{i=1}^3 \langle K_i Y, Z \rangle [J_i, J_s] \in (\mathcal{I}Y + \mathcal{I}Z) \wedge \mathbb{R}^n.$$

Take $Y, Z \in \mathcal{I}v$ in (24). Then by the J^2 -property, $\mathcal{I}Y + \mathcal{I}Z = \mathcal{I}v$ and $[V(Y, Z), J_s] \in \mathcal{I}v \wedge \mathcal{I}v$, so (24) simplifies to $\sum_{i \neq s} \varepsilon_{is} \langle K_i Y, Z \rangle J_j \in \mathcal{I}v \wedge \mathbb{R}^n$, where $\{i, j, s\} = \{1, 2, 3\}$. Projecting this to the subspace $(\mathcal{I}v)^\perp \wedge (\mathcal{I}v)^\perp \subset \mathfrak{o}(n)$ (with respect to the standard inner product on $\mathfrak{o}(n)$) and using the fact that $(\mathcal{I}v)^\perp$ is \mathcal{J} -invariant and $n \geq 8$, we get $\langle K_i Y, Z \rangle = 0$, for all $i = 1, 2, 3$ and all $Y, Z \in \mathcal{I}v$. Introduce the operators $\hat{J}_i = \pi_{\mathcal{I}v} J_i \pi_{\mathcal{I}v}$, $\hat{H} = \pi_{\mathcal{I}v} H \pi_{\mathcal{I}v}$ on $\mathcal{I}v$. As $\mathcal{I}v$ is \mathcal{J} -invariant, the \hat{J}_i 's are anticommuting almost Hermitian structures on $\mathcal{I}v$. Then the condition $\langle K_i Y, Z \rangle = 0$, $Y, Z \in \mathcal{I}v$, and (23) imply

$$(4\omega_i + \lambda_i)v \wedge \hat{J}_i v + 4\varepsilon_{jk}(\omega_j + \omega_k)\hat{J}_j v \wedge \hat{J}_k v + \lambda_i(48C + \|v\|^2)\hat{J}_i + \hat{J}_i \hat{H} + \hat{H} \hat{J}_i = 0.$$

Multiplying by \hat{J}_i and taking the trace we obtain $4\|v\|^2(\omega_i + \omega_j + \omega_k) + \lambda_i(96C + 3\|v\|^2) + \text{Tr} \hat{H} = 0$, where $\{i, j, k\} = \{1, 2, 3\}$, so $\lambda_i(96C + 3\|v\|^2)$ does not depend on $i = 1, 2, 3$. As the λ_i 's are pairwise distinct (otherwise the condition $\sum_{i=1}^3 \eta_i^{-1} = 0$ from (20) is violated), we get $\|v\|^2 = -32C$.

Now take $Y, Z \perp \mathcal{I}v$ in (24). Projecting to $\mathcal{I}v \wedge \mathcal{I}v$ and using the fact that $\mathcal{I}v \wedge \mathcal{I}v$ is $\text{ad}_{\mathcal{J}}$ -invariant we obtain that the operator $V(Y, Z) + \sum_{i=1}^3 \langle K_i Y, Z \rangle \hat{J}_i$ on $\mathcal{I}v$ commutes with every \hat{J}_s . The centralizer of the set $\{\hat{J}_1, \hat{J}_2, \hat{J}_3\}$ in the Lie algebra $\mathfrak{o}(4) = \mathfrak{o}(\mathcal{I}v)$ is the three-dimensional subalgebra spanned by $v \wedge \hat{J}_i v - \varepsilon_{jk} \hat{J}_j v \wedge \hat{J}_k v$, $\{i, j, k\} = \{1, 2, 3\}$ ("the right multiplication by the imaginary quaternions"). Substituting $V(Y, Z)$ from (22) and using the fact that $\hat{J}_i = \|v\|^{-2}(v \wedge \hat{J}_i v + \varepsilon_{jk} \hat{J}_j v \wedge \hat{J}_k v)$ we obtain that the operator $V(Y, Z) + \sum_{i=1}^3 \langle K_i Y, Z \rangle \hat{J}_i$ commutes with all the \hat{J}_s 's, for $Y, Z \perp \mathcal{I}v$, if and only if $-\frac{1}{2}\langle J_i Z, Y \rangle(\lambda_i^2 + \lambda_j \lambda_k - \lambda_i \lambda_k - \lambda_j \lambda_i) + 2\|v\|^{-2}\langle K_i Y, Z \rangle = 0$, for all $i = 1, 2, 3$. Substituting the λ_i 's from (21) and $\langle K_i Y, Z \rangle$ from (23) and taking into account that $\|v\|^2 = -32C$, which is shown above, we obtain $\langle (J_i H + H J_i - 32C \lambda_i J_i) Y, Z \rangle = 0$, for all $Y, Z \perp \mathcal{I}v$ and all $i = 1, 2, 3$. Then $\pi(J_i H + H J_i)\pi = 32C \lambda_i \pi J_i \pi$, where $\pi = \pi_{(\mathcal{I}v)^\perp}$. Multiplying both sides by $\pi J_i \pi$ from the right and using the fact that $[\pi, J_i] = 0$ (as $(\mathcal{I}v)^\perp$ is \mathcal{J} -invariant) we get $\pi(J_i H J_i - H)\pi = -32C \lambda_i \pi$. Taking the traces of the both sides we obtain $-2\text{Tr}(\pi H \pi) = -32C \lambda_i(n - 4)$, which is a contradiction, as $n > 4$ and the λ_i 's are pairwise distinct (which follows from the equation $\sum_{i=1}^3 \eta_i^{-1} = 0$ of (20)). \square

The next lemma shows that the relations similar to (11, 12) of Lemma 5 also hold in all the remaining cases when $n = 8$ (that is, when $\nu \neq 3$ and when $\nu = 3$ and $J_1 J_2 \neq \pm J_3$). As it is shown in Lemma 4, in all these cases the Weyl tensor has a smooth $\text{Cliff}(7)$ -structure in a neighbourhood \mathcal{U} of every point $x \in M'$. Moreover, by assertion 2 of Lemma 2, that $\text{Cliff}(7)$ -structure is an "almost Hermitian octonion structure", in the following sense. For every $y \in \mathcal{U}$, we can identify $\mathbb{R}^8 = T_y M^8$ with \mathbb{O} and of \mathbb{R}^7 with $\mathbb{O}' = 1^\perp$ via linear isometries ι_1, ι_2 respectively in such a way that the orthogonal multiplication (2) defined by $\text{Cliff}(7)$ has the form (5): $J_u X = Xu$, for every $X \in \mathbb{R}^8 = \mathbb{O}$, $u \in \mathbb{O}'$.

Lemma 6. *Let $x \in M' \subset M^8$ and let \mathcal{U} be the neighbourhood of x defined in Lemma 4. For every point $y \in \mathcal{U}$, identify $\mathbb{R}^8 = T_y M^8$ with \mathbb{O} via a linear isometry in such a way that the Clifford structure $\text{Cliff}(7)$ on \mathbb{R}^8 is given by (5). Then there exist $m, t, b_{ij} \in \mathbb{R}^8 = \mathbb{O}$, $i, j = 1, \dots, 7$, such that for all $X, U \in \mathbb{R}^8 = \mathbb{O}$,*

$$(25a) \quad (\nabla_U J_i)X = \sum_{j=1}^7 \langle b_{ij}, U \rangle X e_j + (X(U^* m) - \langle m, U \rangle X) e_i + \langle m, U e_i \rangle X,$$

$$(25b) \quad b_{ij} + b_{ji} = 0,$$

$$(25c) \quad (\nabla_X \rho)U - (\nabla_U \rho)X = \frac{3}{4}(X \wedge U)t + 2 \sum_{i=1}^7 \eta_i (\langle m e_i, U \rangle X e_i - \langle m e_i, X \rangle U e_i + 2\langle X e_i, U \rangle m e_i),$$

$$(25d) \quad \nabla \eta_i = -4\eta_i m - \frac{1}{2}t.$$

Proof. In the proof we use standard identities of the octonion arithmetic (some of them are given in Subsection 2.2).

By [N2, Lemma 7], for the Clifford structure $\text{Cliff}(7)$ given by (5), there exist $b_{ij} \in \mathbb{R}^8$, $i, j = 1, \dots, 7$, satisfying (25b) and an $(\mathbb{R}-)$ linear operator $A : \mathbb{O} \rightarrow \mathbb{O}'$ such that for all $X, U \in \mathbb{R}^8 = \mathbb{O}$,

$$(26) \quad (\nabla_U J_i)X = \sum_{j=1}^7 \langle b_{ij}, U \rangle X e_j + (X \cdot AU) e_i + \langle AU, e_i \rangle X.$$

Equation (8) is a polynomial equation in 24 real variables, the coordinates of the vectors $X, Y, U \in \mathbb{R}^8$. It still holds, if we allow X, Y, U to be complex and extend the tensors $J_i, \nabla J_i$ and $\langle \cdot, \cdot \rangle$ to \mathbb{C}^8 by the complex linearity. The complexified inner product $\langle \cdot, \cdot \rangle$ takes values in \mathbb{C} and is a nonsingular quadratic form on \mathbb{C}^8 . Moreover, equation (5) is still true, if we identify \mathbb{C}^8 with the bioctonion algebra $\mathbb{O} \otimes \mathbb{C}$, and \mathbb{C}^7 with $1^\perp = \mathbb{O}' \otimes \mathbb{C}$, the orthogonal complement to 1 in $\mathbb{O} \otimes \mathbb{C}$.

Let $Y \in \mathbb{O} \otimes \mathbb{C}$ be a nonzero isotropic vector (that is, $Y^* Y = 0$) and let $\mathcal{J}_{\mathbb{C}} Y = \text{Span}_{\mathbb{C}}(J_1 Y, \dots, J_7 Y)$. Then $Y \in \mathcal{J}_{\mathbb{C}} Y$ and the space $\mathcal{J}_{\mathbb{C}} Y$ is isotropic: the inner product of any two vectors from $\mathcal{J}_{\mathbb{C}} Y$ vanishes. Choose $X, U \in \mathcal{J}_{\mathbb{C}} Y$ and take the inner product of the complexified equation (8) with X . As X, Y and U are mutually orthogonal, we get (9), which further simplifies to $\sum_{i=1}^7 \eta_i \langle J_i X, U \rangle \langle (\nabla_Y J_i) Y, X \rangle = 0$, as $\|X\|^2 = \|Y\|^2 = \langle J_i Y, X \rangle = \langle J_i X, U \rangle = 0$. Using (26) we obtain $\sum_{i=1}^7 \eta_i \langle J_i X, U \rangle \langle (Y \cdot AY) e_i, X \rangle = 0$, for all isotropic vectors Y and for all $X, U \in \mathcal{J}_{\mathbb{C}} Y$. It follows that $Y \cdot AY \perp \sum_{i=1}^7 \eta_i \langle J_i X, U \rangle X e_i$, for all $X, U \in \mathcal{J}_{\mathbb{C}} Y$. As $Y \cdot AY = J_{AY} Y \in \mathcal{J}_{\mathbb{C}} Y$ and $\mathcal{J}_{\mathbb{C}} Y$ is isotropic, we get $Y \cdot AY \perp \mathcal{J}_{\mathbb{C}} Y$, so $Y \cdot AY \perp \mathcal{J}_{\mathbb{C}} Y + \text{Span}_{\mathbb{C}}(\{\sum_{i=1}^7 \eta_i \langle J_i X, U \rangle J_i X \mid X, U \in \mathcal{J}_{\mathbb{C}} Y\})$. Following the arguments in the proof of [N2, Lemma 8] starting with formula (29), we obtain that $AU = U^* m - \langle U, m \rangle 1$, for some $m \in \mathbb{O}$. Then equation (25a) follows from (26).

To prove (25c) and (25d), introduce the vectors $f_{ij} \in \mathbb{R}^8$, $i, j = 1, \dots, 8$, and the quadratic map $T : \mathbb{R}^8 \rightarrow \mathbb{R}^8$ (similar to the map Q of (10)) by

$$(27) \quad f_{ij} = (\eta_i - \eta_j) b_{ij} + \delta_{ij} (\nabla \eta_i - 2\eta_i m),$$

$$(28) \quad \langle T(X), U \rangle = \frac{1}{3} \langle (\nabla_X \rho) U - (\nabla_U \rho) X, X \rangle - \sum_{i=1}^7 \eta_i \langle m e_i, X \rangle \langle X e_i, U \rangle.$$

Note that $f_{ij} = f_{ji}$ and $\langle T(X), X \rangle = 0$. Take X, Y, U to be mutually orthogonal vectors in \mathbb{R}^8 . By (25a) and (25b), $\langle (\nabla_U J_i) X, Y \rangle = \sum_{j=1}^7 \langle b_{ij}, U \rangle \langle X e_j, Y \rangle - \langle m, U \rangle \langle X e_i, Y \rangle + \langle (X(U^* m)) e_i, Y \rangle = \sum_{j=1}^7 \langle b_{ij} - \delta_{ij} m, U \rangle \langle X e_j, Y \rangle + \langle m((e_i Y^*) X), U \rangle$, so every term on the left-hand side of (9) can be written as the inner product of a vector depending on X and Y by U . As $U \perp X, Y$ is arbitrary, we find after substituting (5) and (25a) into (9) and rearranging the terms:

$$\begin{aligned} & \|X\|^2 T(Y) + \|Y\|^2 T(X) + 2 \sum_{i=1}^7 \eta_i \langle Y e_i, X \rangle (m((e_i Y^*) X) + (Y(X^* m)) e_i) \\ & + \sum_{i,j=1}^7 \langle Y e_j, X \rangle (\langle f_{ij}, X \rangle Y e_i - \langle f_{ij}, Y \rangle X e_i) - \sum_{i,j=1}^7 \langle Y e_i, X \rangle \langle Y e_j, X \rangle f_{ij} \in \text{Span}(X, Y), \end{aligned}$$

for all $X \perp Y$ (where we used the fact that $(X(Y^* m)) e_i = -(Y(X^* m)) e_i$, as $X \perp Y$). Taking the inner products with X and with Y we obtain

$$\begin{aligned} & \|X\|^2 T(Y) + \|Y\|^2 T(X) + 2 \sum_{i=1}^7 \eta_i \langle Y e_i, X \rangle (m((e_i Y^*) X) + (Y(X^* m)) e_i) \\ & + \sum_{i,j=1}^7 \langle Y e_j, X \rangle (\langle f_{ij}, X \rangle Y e_i - \langle f_{ij}, Y \rangle X e_i) - \sum_{i,j=1}^7 \langle Y e_i, X \rangle \langle Y e_j, X \rangle f_{ij} \\ & = \langle T(Y), X \rangle X + \langle T(X), Y \rangle Y, \end{aligned}$$

for all $X \perp Y$. Taking $X = Yu$, $u = \sum_{i=1}^7 u_i e_i \in \mathbb{O}'$ and regrouping the terms we obtain

$$\begin{aligned} & \|u\|^2 T(Y) + T(Yu) + 2 \sum_{i=1}^7 \eta_i u_i (2 \langle Y, m e_i \rangle Yu - 2 \langle Yu, m e_i \rangle Y + 2 \|Y\|^2 (m u) e_i) \\ (29) \quad & + \sum_{i,j=1}^7 u_j (\langle f_{ij} + 8 \delta_{ij} \eta_i m, Yu \rangle Y e_i - \langle f_{ij} + 8 \delta_{ij} \eta_i m, Y \rangle (Yu) e_i) - \sum_{i,j=1}^7 \|Y\|^2 u_i u_j f_{ij} \\ & = \|Y\|^{-2} \langle T(Y), Yu \rangle Yu + \|Y\|^{-2} \langle T(Yu), Y \rangle Y, \end{aligned}$$

where we used $m((e_i Y^*) X) + (Y(X^* m)) e_i = 2 \langle Y, m e_i \rangle Yu - 2 \langle Yu, m e_i \rangle Y + 4 \langle Yu, m \rangle Y e_i - 4 \langle Y, m \rangle (Yu) e_i + 2 \|Y\|^2 (m u) e_i$, which follows from $m((e_i Y^*) X) = (Y(X^* m)) e_i - 2 \langle m, Y e_i \rangle X - 2 \langle X, m e_i \rangle Y$, for all X, Y ,

and $(Y(X^*m))e_i = -2\langle Y, m \rangle(Yu)e_i - 2\langle Y, mu \rangle Ye_i + \|Y\|^2(mu)e_i$, for $X = Yu$, $u \perp 1$. By assertion 1 of Lemma 3 (with $\nu = 1$ and $\mathcal{I}Y = \text{Span}(Y, Yu)$) we obtain that both coefficients on the right-hand side of (29), $\|Y\|^{-2}\langle T(Y), Yu \rangle$ and $\|Y\|^{-2}\langle T(Yu), Y \rangle$, are linear forms of $Y \in \mathbb{R}^8$, for every $u \in \mathbb{O}'$. As $\langle T(Y), Y \rangle = 0$, this implies that there exists an (\mathbb{R}) -linear operator $C : \mathbb{O} \rightarrow \mathbb{O}'$ such that $\|Y\|^{-2}Y^*T(Y) = CY$, so $T(Y) = Y \cdot CY$, for all $Y \in \mathbb{O}$. Substituting this to (29) and rearranging the terms we obtain

$$(30) \quad \begin{aligned} & (Yu) \left(C(Yu) - \sum_{i,j=1}^7 u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Y \rangle e_i \right) \\ & + Y \left(\|u\|^2 CY + 4 \sum_{i=1}^7 \eta_i u_i (\langle Y, me_i \rangle u - \langle Yu, me_i \rangle 1 + Y^*((mu)e_i)) \right. \\ & \left. + \sum_{i,j=1}^7 u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Yu \rangle e_i - \sum_{i,j=1}^7 u_i u_j Y^* f_{ij} - \langle CY, u \rangle u + \langle C(Yu), u \rangle 1 \right) = 0, \end{aligned}$$

The left-hand side of (30) has the form $(Yu)L(Y, u) + YF(Y, u)$, where $L(Y, u)$ and $F(Y, u)$ are (\mathbb{R}) -linear operator on \mathbb{O} , for every $u \in \mathbb{O}'$. By [N2, Lemma 6], for every unit octonion $u \in \mathbb{O}'$, $L(Y, u) = \langle a(u), Y \rangle 1 + \langle t(u), Y \rangle u + Y^*p(u)$, for some functions $a, t, p : S^6 \subset \mathbb{O}' \rightarrow \mathbb{O}$. Extending a, t, p by homogeneity (of degree 1, 0, 1 respectively) to \mathbb{O}' we obtain $C(Yu) - \sum_{i,j=1}^7 u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Y \rangle e_i = \langle a(u), Y \rangle 1 + \langle t(u), Y \rangle u + Y^*p(u)$, for all $u \in \mathbb{O}'$. Moreover, $p(u) = -a(u)$, as $C(Y) \perp 1$. By the linearity of the left-hand side by u , we get $\langle a(u_1 + u_2) - a(u_1) - a(u_2), Y \rangle 1 + \langle t(u_1 + u_2) - t(u_1), Y \rangle u_1 + \langle t(u_1 + u_2) - t(u_2), Y \rangle u_2 + Y^*(a(u_1 + u_2) - a(u_1) - a(u_2)) = 0$, for all $u_1, u_2 \in \mathbb{O}'$. Then $Y^*(a(u_1 + u_2) - a(u_1) - a(u_2)) \in \text{Span}(1, u_1, u_2)$, for all $Y \in \mathbb{O}$, which is only possible when $a(u)$ is linear, that is $a(u) = Bu$, for some (\mathbb{R}) -linear operator $B : \mathbb{O}' \rightarrow \mathbb{O}$. It follows that $t(u_1 + u_2) = t(u_1) = t(u_2)$, that is, $t \in \mathbb{O}$ is a constant. So $C(Yu) = \sum_{i,j=1}^7 u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Y \rangle e_i + \langle Bu, Y \rangle 1 + \langle t, Y \rangle u - Y^*Bu$. Taking the inner product of the both sides with $v \in \mathbb{O}'$ and subtracting from the resulting equation the same equation with u and v interchanged we obtain $\langle C(Yu), v \rangle - \langle C(Yv), u \rangle = \langle Bv, Yu \rangle - \langle Bu, Yv \rangle$, since $f_{ij} = f_{ji}$ by (27). It follows that $\langle C^t v - Bv, Yu \rangle = \langle C^t u - Bu, Yv \rangle$, where C^t is the operator adjoint to C . Now taking $u \perp v$ and $Y = uv$ we get $\|u\|^2 \langle C^t v - Bv, v \rangle = -\|v\|^2 \langle C^t u - Bu, u \rangle$, which implies $C = B^t$. Then from the above, $\langle C(Yu), e_i \rangle = \sum_{j=1}^7 u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Y \rangle + \langle t, Y \rangle u_i - \langle Bu, Ye_i \rangle = \langle Be_i, Yu \rangle$, so $\sum_{j=1}^7 u_j (f_{ij} + \delta_{ij}(8\eta_i m + t)) + (Bu)e_i + (Be_i)u = 0$. Therefore

$$(31) \quad T(Y) = Y \cdot CY = Y \cdot B^t Y, \quad f_{ij} = -\delta_{ij}(8\eta_i m + t) - (Be_i)e_j - (Be_j)e_i.$$

Substituting (31) to (30) and simplifying we obtain $-\langle Lu \cdot u, Y \rangle Y - \langle Lu, Y \rangle Yu + \|Y\|^2 Lu \cdot u = 0$, where $Lu = 4Bu - tu - 4 \sum_{i=1}^7 \eta_i u_i me_i$. Taking $Y \perp Lu$, $Lu \cdot u$ we get $Lu = 0$, so

$$(32) \quad Bu = \frac{1}{4}tu + \sum_{i=1}^7 \eta_i u_i me_i.$$

Substituting (32) to the first equation of (31) and then to (28) and simplifying we obtain that for arbitrary $X, U \in \mathbb{O}$, $\langle (\nabla_X \rho)U - (\nabla_U \rho)X, X \rangle = \frac{3}{4}(\langle t, X \rangle \langle X, U \rangle - \|X\|^2 \langle t, U \rangle) + 6 \sum_{i=1}^7 \eta_i \langle X e_i, U \rangle \langle me_i, X \rangle$. Polarizing this equation we get

$$\begin{aligned} \langle (\nabla_Y \rho)U - (\nabla_U \rho)Y, X \rangle + \langle (\nabla_X \rho)U - (\nabla_U \rho)X, Y \rangle &= \frac{3}{4}(\langle t, X \rangle \langle Y, U \rangle + \langle t, Y \rangle \langle X, U \rangle - 2\langle X, Y \rangle \langle t, U \rangle) \\ &+ 6 \sum_{i=1}^7 \eta_i (\langle X e_i, U \rangle \langle me_i, Y \rangle + \langle Y e_i, U \rangle \langle me_i, X \rangle). \end{aligned}$$

Subtracting the same equation, with X and U interchanged and using the fact that ρ is symmetric we get (25c). The second equation of (31) and (32) give $f_{ii} = -6\eta_i m - \frac{1}{2}t$, which by (27) implies (25d). \square

Lemma 7. *In the assumptions of Theorem 3, let $x \in M'$, where $M' \subset M^n$ is defined in Lemma 4. Then there exists a neighbourhood $\mathcal{U} = \mathcal{U}(x)$ and a smooth metric on \mathcal{U} conformally equivalent to the original metric whose curvature tensor has the form (7), with ρ a multiple of the identity.*

Proof. Let $x \in M'$ and let \mathcal{U} be the neighbourhood of x on which the Weyl tensor has the smooth Clifford structure defined in Lemma 4. We can assume that $\nu > 0$, as in the case of a $\text{Cliff}(0)$ -structure, the curvature tensor given by (7) has the form $R(X, Y)Z = \langle X, Z \rangle \rho Y + \langle \rho X, Z \rangle Y - \langle Y, Z \rangle \rho X - \langle \rho Y, Z \rangle X$, so the Weyl tensor vanishes. Then the metric on \mathcal{U} is locally conformally flat, that is, is conformally equivalent to a one with $\rho = 0$.

If $n = 8$, $\nu = 7$, and all the η_i 's at x are equal, then they are equal at some neighbourhood of x (by definition of M'). By Remark 3, we can replace ρ by $\rho + \frac{3}{2}\eta_1 \text{id}$ and η_i by $0 = \eta_i - \eta_1$ in (7) arriving at the case $\nu = 0$ considered above.

For the remaining part of the proof, we will assume that in the case $n = 8$, $\nu = 7$, at least two of the η_i 's at x are different; up to relabelling, let $\eta_1 \neq \eta_2$ at x , and also on a neighbourhood of x (replace \mathcal{U} by a smaller neighbourhood, if necessary). Let f be a smooth function on \mathcal{U} and let $\langle \cdot, \cdot \rangle' = e^f \langle \cdot, \cdot \rangle$. Then $W' = W$, $J'_i = J_i$, $\eta'_i = e^{-f} \eta_i$ and, on functions, $\nabla' = e^{-f} \nabla$, where we use the dash for the objects associated to metric $\langle \cdot, \cdot \rangle'$. Moreover, the curvature tensor R' still has the form (7), and all the identities of Lemma 5 and of Lemma 6 remain valid.

In the cases considered in Lemma 5, the ratios η_i/η_1 are constant, as it follows from (11d, 12c). In particular, taking $f = \ln |\eta_1|$ we obtain that η'_1 is a constant, so all the η'_i are constant, $m'_i = 0$ by (11d), so $(\nabla'_Y \rho')U - (\nabla'_U \rho')Y = 0$ by (12a). In the case $n = 8$, $\nu = 7$ (Lemma 6), take $f = \ln |\eta_1 - \eta_2|$. Then by (25d), $\nabla f = -4m$ and $\nabla' \eta'_i = -\frac{1}{2}e^{-2f} t$ which implies $m' = -\frac{1}{4} \nabla' \ln |\eta'_1 - \eta'_2| = 0$, $t' = e^{-2f} t$, again by (25d) for the metric $\langle \cdot, \cdot \rangle'$. Then by (25c), $(\nabla'_X \rho')U - (\nabla'_U \rho')X = \frac{3}{4}(X \wedge' U)t'$. By Remark 3, we can replace ρ' by $\tilde{\rho} = \rho' + \frac{3}{2}(\eta'_1 + C) \text{id}$ and η'_i by $\tilde{\eta}_i = \eta'_i - (\eta'_1 + C)$ without changing the curvature tensor R' given by (7) (C is a constant chosen in such a way that $\tilde{\eta}_i \neq 0$ anywhere on \mathcal{U}). Then by (25c) and (25d) for the metric $\langle \cdot, \cdot \rangle'$, $(\nabla'_X \tilde{\rho})U - (\nabla'_U \tilde{\rho})X = 0$.

Dropping the dashes and the tildes, we obtain that, up to a conformal smooth change of the metric on \mathcal{U} , the curvature tensor has the form (7), with ρ satisfying the identity

$$(\nabla_Y \rho)X = (\nabla_X \rho)Y,$$

for all X, Y , that is, with ρ being a symmetric *Codazzi tensor*.

Then by [DS, Theorem 1], at every point of \mathcal{U} , for any three eigenspaces $E_\beta, E_\gamma, E_\alpha$ of ρ , with $\alpha \notin \{\beta, \gamma\}$, the curvature tensor satisfies $R(X, Y)Z = 0$, for all $X \in E_\beta$, $Y \in E_\gamma$, $Z \in E_\alpha$. It then follows from (7) that

$$(33) \quad \sum_{i=1}^{\nu} \eta_i (2\langle J_i X, Y \rangle J_i Z + \langle J_i Z, Y \rangle J_i X - \langle J_i Z, X \rangle J_i Y) = 0, \\ \text{for all } X \in E_\beta, Y \in E_\gamma, Z \in E_\alpha, \quad \alpha \notin \{\beta, \gamma\}.$$

Suppose ρ is not a multiple of the identity. Let E_1, \dots, E_p , $p \geq 2$, be the eigenspaces of ρ . If $p > 2$, denote $E'_1 = E_1$, $E'_2 = E_2 \oplus \dots \oplus E_p$. Then by linearity, (33) holds for any $X, Y \in E'_\alpha$, $Z \in E'_\beta$, such that $\{\alpha, \beta\} = \{1, 2\}$. Hence to prove the lemma it suffices to show that (33) leads to a contradiction, in the assumption $p = 2$. For the rest of the proof, suppose that $p = 2$. Denote $\dim E_\alpha = d_\alpha$, $d_1 \leq d_2$.

Choose $Z \in E_\alpha$, $X, Y \in E_\beta$, $\alpha \neq \beta$, and take the inner product of (33) with X . We get $\sum_{i=1}^{\nu} \eta_i \langle J_i X, Y \rangle \langle J_i X, Z \rangle = 0$. It follows that for every $X \in E_\alpha$, the subspaces E_1 and E_2 are invariant subspaces of the symmetric operator $\hat{R}_X \in \text{End}(\mathbb{R}^n)$ defined by $\hat{R}_X Y = \sum_{i=1}^{\nu} \eta_i \langle J_i X, Y \rangle J_i X$. So \hat{R}_X commutes with the orthogonal projections $\pi_\beta : \mathbb{R}^n \rightarrow E_\beta$, $\beta = 1, 2$. Then for all $\alpha, \beta = 1, 2$ (α and β can be equal), all $X \in E_\alpha$ and all $Y \in \mathbb{R}^n$, $\sum_{i=1}^{\nu} \eta_i \langle J_i X, \pi_\beta Y \rangle J_i X = \sum_{i=1}^{\nu} \eta_i \langle J_i X, Y \rangle \pi_\beta J_i X$. Taking $Y = J_j X$ we get that $\pi_\beta J_j X \subset \mathcal{J}X$, that is $\pi_\beta \mathcal{J}X \subset \mathcal{J}X$, for all $X \in E_\alpha$, $\alpha, \beta = 1, 2$. As $\pi_1 + \pi_2 = \text{id}$, we obtain $\mathcal{J}X \subset \pi_1 \mathcal{J}X \oplus \pi_2 \mathcal{J}X \subset \mathcal{J}X$, hence $\mathcal{J}X = \pi_1 \mathcal{J}X \oplus \pi_2 \mathcal{J}X$. As every function $f_{\alpha\beta} : E_\alpha \rightarrow \mathbb{Z}$, $\alpha, \beta = 1, 2$, defined by $f_{\alpha\beta}(X) = \dim \pi_\beta \mathcal{J}X$, $X \in E_\alpha$, is lower semi-continuous, and $f_{\alpha 1}(X) + f_{\alpha 2}(X) = \nu$ for all nonzero $X \in E_\alpha$, there exist constants $c_{\alpha\beta}$, with $c_{\alpha 1} + c_{\alpha 2} = \nu$, such that $\dim \pi_\beta \mathcal{J}X = c_{\alpha\beta}$, for all $\alpha, \beta = 1, 2$ and all nonzero $X \in E_\alpha$.

Let $X, Y \in E_\alpha$, $Z \in E_\beta$, $\beta \neq \alpha$. Taking the inner product of (33) with $J_j Z$, $j = 1, \dots, \nu$, we get

$$2\eta_j \langle J_j X, Y \rangle \|Z\|^2 = \sum_{i \neq j} \eta_i (\langle J_i Z, X \rangle \langle J_i Y, J_j Z \rangle - \langle J_i Z, Y \rangle \langle J_i X, J_j Z \rangle).$$

As $\langle J_i Z, X \rangle = \langle J_i \pi_\beta Z, X \rangle = -\langle Z, \pi_\beta J_i X \rangle$ (and similarly for $\langle J_i Z, Y \rangle$), the right-hand side, viewed as a quadratic form of $Z \in E_\beta$, vanishes for all $Z \in (\pi_\beta \mathcal{J}X)^\perp \cap (\pi_\beta \mathcal{J}Y)^\perp$, that is, on a subspace of dimension at least $d_\beta - 2c_{\alpha\beta}$. So for $\alpha \neq \beta$, either $2c_{\alpha\beta} \geq d_\beta$, or $\mathcal{J}E_\alpha \perp E_\alpha$, that is, $\pi_\beta \mathcal{J}X = \mathcal{J}X$, for all $X \in E_\alpha$, so $c_{\alpha\beta} = \nu$.

Similarly, if $Z \in E_\alpha$, $X, Y \in E_\beta$, $\beta \neq \alpha$, the inner product of (33) with $J_j X$, $j = 1, \dots, \nu$, gives

$$\eta_j \langle J_j Z, Y \rangle \|X\|^2 = \sum_{i=1}^{\nu} \eta_i (-2\langle J_i X, Y \rangle \langle J_i Z, J_j X \rangle + \langle J_i Z, X \rangle \langle J_i Y, J_j X \rangle).$$

As $\langle J_i X, Y \rangle = -\langle X, \pi_\beta J_i Y \rangle$, $\langle J_i Z, X \rangle = -\langle X, \pi_\beta J_i Z \rangle$, the right-hand side, viewed as a quadratic form of $X \in E_\beta$, vanishes on the subspace $(\pi_\beta \mathcal{J}Y)^\perp \cap (\pi_\beta \mathcal{J}Z)^\perp$ whose dimension is at least $d_\beta - c_{\alpha\beta} - c_{\beta\beta}$. We obtain that for $\alpha \neq \beta$, either $c_{\alpha\beta} + c_{\beta\beta} \geq d_\beta$, or $\mathcal{J}E_\alpha \perp E_\beta$, that is, $\pi_\beta \mathcal{J}Z = 0$, for all $Z \in E_\alpha$, so $c_{\alpha\beta} = 0$. As the equation $c_{\alpha\beta} = 0$ contradicts both $2c_{\alpha\beta} \geq d_\beta$ and $c_{\alpha\beta} = \nu$ (as $\nu > 0$), we must have $c_{\alpha\beta} + c_{\beta\beta} \geq d_\beta$. Then $2\nu = \sum_{\alpha\beta} c_{\alpha\beta} \geq d_1 + d_2 = n$.

This proves the lemma in all the cases when $2\nu < n$, that is, in all the cases except for $n = 8$, $\nu \geq 4$ (as it follows from Lemma 1).

Consider the case $n = 8$. We identify \mathbb{R}^8 with \mathbb{O} and assume that the J_i 's act as in (5). Let $D : \mathbb{O} \rightarrow \mathbb{O}$ be the symmetric operator defined by $D1 = 0$, $De_i = \eta_i e_i$. By (4), condition (33) still holds if we replace D by $D + c \operatorname{Im}$, where Im is the operator of taking the imaginary part of an octonion. So we can assume that the eigenvalue of the maximal multiplicity of $D|_{\mathbb{O}'}$ is zero (one of them, if there are more than one). Then in (33), $\nu = \operatorname{rk} D$. By construction, $\nu \leq 6$, and we only need to consider the cases when $\nu \geq 4$, as it is shown above.

By (5), $\langle J_i X, Y \rangle J_i Z = \langle X e_i, Y \rangle Z e_i = \langle e_i, X^* Y \rangle Z e_i$, so $\sum_{i=1}^\nu \eta_i \langle J_i X, Y \rangle J_i Z = \sum_{i=1}^\nu \eta_i \langle e_i, X^* Y \rangle Z e_i = \sum_{i=1}^7 \langle D e_i, X^* Y \rangle Z e_i = Z D(X^* Y)$, as D is symmetric and $D1 = 0$. Then (33) can be rewritten as

$$(34) \quad 2ZD(X^*Y) + XD(Z^*Y) - YD(Z^*X) = 0, \quad \text{for all } X, Y \in E_\beta, Z \in E_\alpha, \alpha \neq \beta.$$

Taking the inner product of (34) with X (and using the fact that D is symmetric, $D1 = 0$ and $Y^*X = 2\langle X, Y \rangle 1 - X^*Y$) we obtain $\langle D(X^*Y), X^*Z \rangle = 0$. It follows that for every $X \in E_\beta$, the subspaces E_1 and E_2 are invariant subspaces of the symmetric operator $L_X D L_X^t$, where $L_X : \mathbb{O} \rightarrow \mathbb{O}$ is the left multiplication by X (note that $L_{X^*} = L_X^t$ and that $L_X D L_X^t$ coincides with the operator \hat{R}_X introduced above). So $L_X D L_X^t$ commutes with the both orthogonal projections $\pi_\alpha : \mathbb{R}^8 \rightarrow E_\alpha$, $\alpha = 1, 2$. It follows that for every α, β (not necessarily distinct) and every $X \in E_\beta$, the operator D commutes with $L_X^t \pi_\alpha L_X = \|X\|^2 \pi_{X^* E_\alpha}$, that is,

$$(35) \quad \text{the space } X^* E_\alpha \text{ is an invariant subspace of } D, \text{ for all } \alpha, \beta, \text{ and all } X \in E_\beta.$$

Consider all the possible cases for the dimensions d_α of the subspaces E_α .

Let $(d_1, d_2) = (1, 7)$, and let u be a nonzero vector in E_1 . Then by (35), every line spanned by X^*u , $X \perp u$ (that is, every line in \mathbb{O}') is an invariant subspace of D . It follows that $D|_{\mathbb{O}'}$ is a multiple of the identity, which is a contradiction, as $\operatorname{rk} D = \nu$, $4 \leq \nu \leq 6$.

Let $(d_1, d_2) = (2, 6)$, and let $E_1 = \operatorname{Span}(u, ue)$, $e \in \mathbb{O}'$, $\|e\| = \|u\| = 1$. Then $E_2 = uL$, where $L = \operatorname{Span}(1, e)^\perp$. By (35) with $E_\alpha = E_1$ and $X = uU^* = -uU \in E_2$, $U \in L$, every two-plane $\operatorname{Span}(U, (Uu^*)(ue))$, $U \in L$, is an invariant subspace of D . Note that $(Uu^*)(ue) \in L$, for all $U \in L$, and moreover, the operator J defined by $JU = (Uu^*)(ue)$ is an almost Hermitian structure on L . Then L is an invariant subspace of D (as the sum of the invariant subspaces $\operatorname{Span}(U, JU)$, $U \in L$) and $JD|_L U \in \operatorname{Span}(U, JU)$, for all $U \in L$ (as $\operatorname{Span}(U, JU)$ is both J - and $D|_L$ -invariant). From assertion 1 of Lemma 3 it follows that the operator $JD|_L$ is a linear combination of $\operatorname{id}|_L$ and J . As D is symmetric and its eigenvalue of the maximal multiplicity is zero, $D|_L = 0$. Then $\nu = \operatorname{rk} D \leq 1$, which is a contradiction.

For the cases $(d_1, d_2) = (3, 5), (4, 4)$, we use the notion of the Cayley plane. A four-dimensional subspace $\mathcal{C} \subset \mathbb{O}$ is called a *Cayley plane*, if for orthonormal octonions $X, Y, Z \in \mathcal{C}$, $X(Y^*Z) \in \mathcal{C}$. This definition coincides with [HL, Definition IV.1.23], if we disregard the orientation. We will need the following properties of the Cayley plane (they can be found in [HL, Section IV] or proved directly):

- (i) A Cayley plane is well-defined; moreover, if $X(Y^*Z) \in \mathcal{C}$ for some triple X, Y, Z of orthonormal octonions in \mathcal{C} , then the same is true for any triple $X, Y, Z \in \mathcal{C}$ (possibly, non-orthonormal).
- (ii) If \mathcal{C} is a Cayley plane, then the subspace $X^*\mathcal{C}$ is the same for all nonzero $X \in \mathcal{C}$; we call this subspace $\mathcal{C}^*\mathcal{C}$.
- (iii) If \mathcal{C} is a Cayley plane, then \mathcal{C}^\perp is also a Cayley plane and $\mathcal{C}^{\perp*}\mathcal{C}^\perp = \mathcal{C}^*\mathcal{C}$. Moreover, for all nonzero $X \in \mathcal{C}^\perp$, the subspace $X^*\mathcal{C}$ is the same and is equal to $(\mathcal{C}^*\mathcal{C})^\perp$.
- (iv) For every nonzero $e \in \mathbb{O}$ and every pair of orthonormal imaginary octonions u, v , the subspace $\mathcal{C} = \operatorname{Span}(e, eu, ev, (eu)v)$ is a Cayley plane; every Cayley plane can be obtained in this way.

Let $(d_1, d_2) = (3, 5)$. Then E_1 is contained in a Cayley plane \mathcal{C} (spanned by E_1 and $X(Y^*Z)$, for some orthonormal vectors $X, Y, Z \in E_1$), so $\mathcal{C}^\perp \subset E_2$. Let U be a unit vector in the orthogonal

complement to \mathcal{C}^\perp in E_2 . Then for every nonzero $X \in \mathcal{C}^\perp$, $X^*E_2 = \mathcal{C}^*\mathcal{C} \oplus \mathbb{R}(X^*U)$, by properties (ii, iii). As for any two invariant subspaces of a symmetric operator, their intersection and the orthogonal complements to it in each of them are also invariant, it follows from (35) that both $\mathcal{C}^*\mathcal{C}$ and every line $\mathbb{R}(X^*U)$, $X \in \mathcal{C}^\perp$, are invariant subspaces of D . Then the restriction of D to the four-dimensional space $(\mathcal{C}^\perp)^*U$ is a multiple of the identity on that space. As the eigenvalue of the maximal multiplicity of D is zero, $\mathbb{R}1 \oplus (\mathcal{C}^\perp)^*U \subset \text{Ker} D$. Then $\nu = \text{rk } D \leq 3$, which is again a contradiction.

Let now $d_1 = d_2 = 4$. First assume that E_1 is not a Cayley plane. Let X_1, X_2 be orthonormal vectors in E_1 . Then $X_1^*E_1 \cap X_2^*E_1 \supset \text{Span}(1, X_1^*X_2)$, as $X_2^*X_1 = -X_1^*X_2$. Moreover, for any unit vector $Y \in X_1^*E_1 \cap X_2^*E_1$ orthogonal to $\text{Span}(1, X_1^*X_2)$ we have $Y = X_1^*X_3 = X_2^*X_4$ for some $X_3, X_4 \in E_1$, $X_3, X_4 \perp X_1, X_2$, which implies $X_2(X_1^*X_3) = X_4 \in E_1$, so E_1 is a Cayley plane by property (i). It follows that $X_1^*E_1 \cap X_2^*E_1 = \text{Span}(1, X_1^*X_2)$. As by (35) both subspaces on the left-hand side are invariant under D and as $\mathbb{R}1$ is an invariant subspace of D , we obtain that every line $\mathbb{R}(X_1^*X_2)$, $X_1, X_2 \in E_1$ is an invariant subspace of D (that is, $X_1^*X_2$ is an eigenvector of D). Then the space $L = \text{Span}(E_1^*E_1)$ lies in an eigenspace of D , so $D|_L$ is a multiple of id_L . If $X_1, X_2, X_3 \in E_1$ are orthonormal, then $X_2^*X_3 \notin X_1^*E_1$, as E_1 is not a Cayley plane. So $\dim L \geq 5$. As the eigenvalue of the maximal multiplicity of D is zero, $\nu = \text{rk } D \leq 3$, a contradiction.

Let again $d_1 = d_2 = 4$, and let E_1 be a Cayley plane. Then $E_2 = (E_1)^\perp$ is also a Cayley plane by property (iii). Moreover, by the same property, $E_1^*E_1 = E_2^*E_2 = V_1$ and $E_1^*E_2 = E_2^*E_1 = V_2$, where V_1, V_2 are mutually orthogonal four-dimensional subspaces of \mathbb{O} , and $1 \in V_1$. From (35), each of the two spaces V_1, V_2 is invariant under D . Let $X, Y \in E_1$, $Z, W \in E_2$, with $X, Z \neq 0$, and let $u = X^{-1}Y$, $v = Z^{-1}W$. As $X^{-1} = \|X\|^{-2}X^*$, $L_{X^{-1}}E_1 = V_1$ by property (ii). Similarly, $L_{Z^{-1}}E_2 = V_1$. Taking the inner product of (34) with W we obtain that for all $X \in E_1$, $Z \in E_2$, $u, v \in V_1$,

$$2\|Z\|^2\|X\|^2\langle Du, v \rangle - \langle D(Z^*(Xu)), Z^*(Xv) \rangle = -\langle D(Z^*X), Z^*((Xu)v) \rangle.$$

The left-hand side is symmetric in u, v . As $(Xu)v = -(Xv)u$, for any $u \perp v$, $u, v \perp 1$, we obtain $\langle D(Z^*X), Z^*((Xu)v) \rangle = 0$ for all $u, v \in V_1$, $u \perp v$, $u, v \perp 1$, and all $X \in E_1$, $Z \in E_2$. Given any nonzero orthogonal $X, X' \in E_1$, we can find $u, v \in V_1$, $u \perp v$, $u, v \perp 1$, such that $X' = (Xu)v$. To see that note that for every $u \in V_1 = E_1^*E_1$, $Xu \in E_1$ by property (i). As L_X is nonsingular, $L_X(V_1 \cap 1^\perp)$ is a three-dimensional subspace of E_1 . The same is true with X replaced by X' . Therefore, for some $u, v \in V_1 \cap 1^\perp$, $Xu = X'v$, hence $X' = -\|v\|^{-2}(Xu)v$. As $X' \perp X$, we get $\langle X, (Xu)v \rangle = 0$, so $u \perp v$. Thus $\langle D(Z^*X), Z^*X' \rangle = 0$, for any $Z \in E_2$ and any orthogonal $X, X' \in E_1$. As $Z^*E_1 = V_2$, for any nonzero $Z \in E_2$, by properties ii, ii), and the operator L_{Z^*} is orthogonal when $\|Z\| = 1$ we get $\langle Dv_1, v_2 \rangle = 0$, for any two orthogonal vectors $v_1, v_2 \in V_2$. It follows that the restriction of D to its invariant subspace V_2 is a multiple of the identity. As $V_2 \subset \mathbb{O}'$ and the eigenvalue of $D|_{\mathbb{O}'}$ of the maximal multiplicity is zero we obtain that $\mathbb{R}1 \oplus V_2 \subset \text{Ker} D$. Then $\nu = \text{rk } D \leq 3$ which is a contradiction. \square

Remark 4. As it follows from the proof of Lemma 7, the algebraic statement “a symmetric operator satisfying (33) is a multiple of the identity” is valid when $2\nu < n$. In particular, when $n = 16$, it remains true, if we relax the restrictions $\nu \leq 4$ of Theorem 3 to $\nu \neq 8$ (as for $n = 16$, $\nu \leq 8$ by (3)).

Lemma 7 implies Theorem 3 at the generic points. Indeed, by Lemma 7, every $x \in M'$ has a neighbourhood \mathcal{U} which is either conformally flat or is conformally equivalent to a Riemannian manifold whose curvature tensor has the form (7), with ρ being a multiple of the identity, that is, whose curvature tensor has a Clifford structure. It follows from [N1, Theorem 1.2], [N2, Proposition 2] that \mathcal{U} is conformally equivalent to an open subset of one of the five model spaces: the rank-one symmetric spaces $\mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{H}H^{n/4}$, or the Euclidean space.

To prove Theorem 3 in full, we show that, firstly, the same is true for any $x \in M^n$, and secondly, that the model space to a domain of which \mathcal{U} is conformally equivalent is the same, for all $x \in M^n$.

We normalize the standard metric \tilde{g} on each of the spaces $\mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{H}H^{n/4}$ in such a way that the sectional curvature K_σ satisfies $|K_\sigma| \in [1, 4]$. Then the curvature tensor of each of them has a Clifford structure $\text{Cliff}(\nu; J_1, \dots, J_\nu; \varepsilon, \dots, \varepsilon)$, ($\nu + 1$ ε 's), where $\nu = 1, 3$, $\varepsilon = \pm 1$ and the J_i 's are smooth anticommuting almost Hermitian structures, with $J_1 J_2 = \pm J_3$ when $\nu = 3$ and with $\tilde{\nabla}_Z J_i = \sum_{j=1}^m \omega_i^j(Z) J_j$, where ω_i^j are smooth 1-forms with $\omega_i^j + \omega_j^i = 0$, and $\tilde{\nabla}$ is the Levi-Civita

connection for \tilde{g} . Denote the corresponding spaces by $M_{\nu,\varepsilon}$ (and their Weyl tensors, by $W_{\nu,\varepsilon}$), so that

$$M_{1,1} = (\mathbb{C}P^{n/2}, \tilde{g}), \quad M_{1,-1} = (\mathbb{C}H^{n/2}, \tilde{g}), \quad M_{3,1} = (\mathbb{H}P^{n/4}, \tilde{g}), \quad M_{3,-1} = (\mathbb{H}H^{n/4}, \tilde{g}).$$

We start with the following technical lemma:

Lemma 8. *Let $(N^n, \langle \cdot, \cdot \rangle)$ be a smooth Riemannian space locally conformally equivalent to one of the $M_{\nu,\varepsilon}$, so that $\tilde{g} = f\langle \cdot, \cdot \rangle$, for a positive smooth function $f = e^{2\phi} : N^n \rightarrow \mathbb{R}$. Then the curvature tensor R and the Weyl tensor W of $(N^n, \langle \cdot, \cdot \rangle)$ satisfy*

$$\begin{aligned} (36a) \quad R(X, Y) &= (X \wedge KY + KX \wedge Y) + \varepsilon f(X \wedge Y + T(X, Y)), \quad \text{where} \\ T(X, Y) &= \sum_{i=1}^{\nu} (J_i X \wedge J_i Y + 2\langle J_i X, Y \rangle J_i), \quad K = H(\phi) - \nabla \phi \otimes \nabla \phi + \frac{1}{2} \|\nabla \phi\|^2 \text{id}, \\ (36b) \quad W(X, Y) &= W_{\nu,\varepsilon}(X, Y) = \varepsilon f(-\frac{3\nu}{n-1} X \wedge Y + T(X, Y)), \\ (36c) \quad \|W\|^2 &= C_{\nu n} f^2, \quad C_{\nu n} = 6\nu n(n+2)(n-\nu-1)(n-1)^{-1}, \\ (36d) \quad (\nabla_Z W)(X, Y) &= \varepsilon Z f(-\frac{3\nu}{n-1} X \wedge Y + T(X, Y)) \\ &\quad + \frac{1}{2} \varepsilon ([T(X, Y), \nabla f \wedge Z] + T((\nabla f \wedge Z)X, Y) + T(X, (\nabla f \wedge Z)Y)), \end{aligned}$$

where $X \wedge Y$ is the linear operator defined by $(X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X$, $H(\phi)$ is the symmetric operator associated to the Hessian of ϕ , and both ∇ and the norm are computed with respect to $\langle \cdot, \cdot \rangle$.

Proof. The curvature tensor of $M_{\nu,\varepsilon}$ has the form $\tilde{R}(X, Y) = \varepsilon(X \tilde{\wedge} Y + \sum_{i=1}^{\nu} (J_i X \tilde{\wedge} J_i Y + 2\tilde{g}(J_i X, Y) J_i))$, where $(X \tilde{\wedge} Y)Z = \tilde{g}(X, Z)Y - \tilde{g}(Y, Z)X$. Under the conformal change of metric $\tilde{g} = f\langle \cdot, \cdot \rangle = e^{2\phi}\langle \cdot, \cdot \rangle$, the curvature tensor transforms as $\tilde{R}(X, Y) = R(X, Y) - (X \wedge KY + KX \wedge Y)$. As $\tilde{g}(X, Y) = f\langle X, Y \rangle$, $X \tilde{\wedge} Y = f(X \wedge Y)$ and the J_i 's remain anticommuting almost Hermitian structures for $\langle \cdot, \cdot \rangle$, equation (36a) follows.

The fact that the Weyl tensor has the form (36b) follows from the definition; the norm of W can be computed directly using the fact that the J_i 's are orthogonal and that $J_1 J_2 = \pm J_3$ when $\nu = 3$.

From $\tilde{\nabla}_Z J_i = \sum_{j=1}^{\nu} \omega_i^j(Z) J_j$ and $\tilde{\nabla}_Z X = \nabla_Z X + Z\phi X + X\phi Z - \langle X, Z \rangle \phi$, where $\tilde{\nabla}$ is the Levi-Civita connection for \tilde{g} , we get $\nabla_Z J_i = \sum_{j=1}^{\nu} \omega_i^j(Z) J_j + [J_i, \nabla \phi \wedge Z]$ (where we used the fact that $[J_i, X \wedge Y] = J_i X \wedge Y + X \wedge J_i Y$). Then

$$(\nabla_Z T)(X, Y) = [T(X, Y), \nabla \phi \wedge Z] + T((\nabla \phi \wedge Z)X, Y) + T(X, (\nabla \phi \wedge Z)Y),$$

which, together with (36b), proves (36d). \square

For every point $x \in M'$, there exists a neighbourhood \mathcal{U} of x and a positive smooth function $f : \mathcal{U} \rightarrow \mathbb{R}$ such that the Riemannian space $(\mathcal{U}(x), f\langle \cdot, \cdot \rangle)$ is isometric to an open subset of one of the five model spaces $(M_{\nu,\varepsilon} \text{ or } \mathbb{R}^n)$, so at every point $x \in M'$, the Weyl tensor W of M^n either vanishes, or has the form given in (36b). The Jacobi operators associated to the different Weyl tensors $W_{\nu,\varepsilon}$ in (36b) differ by the multiplicities and the signs of the eigenvalues, so every point $x \in M'$ has a neighbourhood conformally equivalent to a domain of exactly one of the model spaces. Moreover, the function $f > 0$ is well-defined at all the points where $W \neq 0$, as $\|W\|^2 = C_{\nu n} f^2$ by (36c).

By continuity, the Weyl tensor W of M^n either has the form $W_{\nu,\varepsilon}$ or vanishes, at every point $x \in M^n$ (as M' is open and dense in M^n , see Lemma 4). Moreover, every point $x \in M^n$, at which the Weyl tensor has the form $W_{\nu,\varepsilon}$, has a neighbourhood, at which the Weyl tensor has the same form. Hence $M^n = M_0 \cup \bigcup_{\alpha} M_{\alpha}$, where $M_0 = \{x : W(x) = 0\}$ is closed, and every M_{α} is a nonempty open connected subset, with $\partial M_{\alpha} \subset M_0$, such that the Weyl tensor has the same form $W_{\nu,\varepsilon} = W_{\nu(\alpha),\varepsilon(\alpha)}$ at every point $x \in M_{\alpha}$. In particular, $M_{\alpha} \subset M'$, for every α , so that each M_{α} is locally conformally equivalent to one of the model spaces $M_{\nu,\varepsilon}$.

If $M = M_0$ or if $M_0 = \emptyset$, the theorem is proved. Otherwise, suppose that $M_0 \neq \emptyset$ and that there exists at least one component M_{α} . Let $y \in \partial M_{\alpha} \subset M_0$ and let $B_{\delta}(y)$ be a small geodesic ball of M centered at y which is strictly geodesically convex (any two points from $B(y)$ can be connected by a unique geodesic segment lying in $B_{\delta}(y)$ and that segment realizes the distance between them). Let $x \in B_{\delta/3}(y) \cap M_{\alpha}$ and let $r = \text{dist}(x, M_0)$. Then the geodesic ball $B = B_r(x)$ lies in M_{α} and is strictly

convex. Moreover, ∂B contains a point $x_0 \in M_0$. Replacing x by the midpoint of the segment $[xx_0]$ and r by $r/2$, if necessary, we can assume that all the points of ∂B , except for x_0 , lie in M_α .

The function f is positive and smooth on $\overline{B} \setminus \{x_0\}$ (that is, on an open subset containing $\overline{B} \setminus \{x_0\}$, but not containing x_0). We are interested in the behavior of $f(x)$, when $x \in B$ approaches x_0 .

Lemma 9. *When $x \rightarrow x_0$, $x \in B$, both f and ∇f have a finite limit. Moreover, $\lim_{x \rightarrow x_0, x \in B} f(x) = 0$.*

Proof. The fact that $\lim_{x \rightarrow x_0, x \in B} f(x) = 0$ follows from (36c) and the fact that $W|_{x_0} = 0$ (as $x_0 \in M_0$).

As the Riemannian space $(B, f\langle \cdot, \cdot \rangle)$ is locally isometric to a rank-one symmetric space $M_{\nu, \varepsilon}$ and is simply connected, there exists a smooth isometric immersion $\iota : (B, f\langle \cdot, \cdot \rangle) \rightarrow M_{\nu, \varepsilon}$. Since f is smooth on $\overline{B} \setminus \{x_0\}$ and $\lim_{x \rightarrow x_0, x \in B} f(x) = 0$, the range of ι is a bounded domain in $M_{\nu, \varepsilon}$. Moreover, as $\lim_{x \rightarrow x_0, x \in B} f(x) = 0$, every sequence of points in B converging to x_0 in the metric $\langle \cdot, \cdot \rangle$ is a Cauchy sequence for the metric $f\langle \cdot, \cdot \rangle$. It follows that there exists a limit $\lim_{x \rightarrow x_0, x \in B} \iota(x) \in M_{\nu, \varepsilon}$. Defining for every $x \in B$ the point $\mathcal{J}_x = \text{Span}(J_1, \dots, J_\nu)$ in the Grassmanian $G(\nu, \bigwedge^2 T_x M^n)$, we find that there exists a limit $\lim_{x \rightarrow x_0, x \in B} \mathcal{J}_x =: \mathcal{J}_{|x_0} \in G(\nu, \bigwedge^2 T_{x_0} M^n)$. In particular, if Z is a continuous vector field on \overline{B} , then there exists a unit continuous vector field Y on \overline{B} such that $Y \perp Z, \mathcal{J}Z$ on B . For such two vector fields, the function $\theta(Y, Z) = \langle \sum_{j=1}^n (\nabla_{E_j} W)(E_j, Y)Y, Z \rangle$ (where E_j is an orthonormal frame on \overline{B}) is well-defined and continuous on \overline{B} . Using (36d) we obtain by a direct computation that at the points of B , $\theta(Y, Z) = \frac{\varepsilon(n-3)}{2(n-1)} \langle (3\nu \nabla f \wedge Y - (n-1)T(\nabla f, Y))Y, Z \rangle = \frac{-3\varepsilon\nu(n-3)}{2(n-1)} \langle \nabla f, Z \rangle$ (where we used the fact that $\|Y\| = 1$ and $Y \perp Z, \mathcal{J}Z$). As $\theta(Y, Z)$ is continuous on \overline{B} , there exists a limit $\lim_{x \rightarrow x_0, x \in B} Zf$. Since Z is an arbitrary continuous vector field on \overline{B} , ∇f has a finite limit when $x \rightarrow x_0$, $x \in B$. \square

As $\lim_{x \rightarrow x_0, x \in B} f(x) = 0$ and the J_i 's are orthogonal, the second term on the right-hand side of equation (36a) tends to 0 when $x \rightarrow x_0$ in B . Therefore the (3,1) tensor field defined by $(X, Y) \rightarrow (X \wedge KY + KX \wedge Y)$ has a finite limit (namely $R|_{x_0}$) when $x \rightarrow x_0$ in B . It follows that the symmetric operator K has a finite limit at x_0 . Computing the trace of K and using the fact that $\phi = \frac{1}{2} \ln f$ we get

$$(37) \quad \Delta u = Fu, \quad \text{where } u = f^{(n-2)/4}, \quad F = \frac{1}{2}(n-2)\text{Tr}K$$

on B . Both functions F and u are smooth on $\overline{B} \setminus \{x_0\}$ and have a finite limit at x_0 . Moreover, $\lim_{x \rightarrow x_0, x \in B} u(x) = 0$ by Lemma 9 and $u(x) > 0$ for $x \in \overline{B} \setminus \{x_0\}$. The domain B is a small geodesic ball, so it satisfies the inner sphere condition (the radii of curvature of the sphere ∂B are uniformly bounded). By the boundary point theorem [F, Section 2.3], the inner directional derivative of u at x_0 (which exists by Lemma 9, if we define $u(x_0) = 0$ by continuity) is positive.

As $\nabla u = \frac{1}{4}(n-2)f^{(n-6)/4}\nabla f$ in B , we arrive at a contradiction with Lemma 9 in all the cases, except for $n = 6$. To finish the proof in that case, we will show that the limit $\lim_{x \rightarrow x_0, x \in B} \nabla f(x)$, which exists by Lemma 9, is zero. When $n = 6$, we have $\nu = 1$ by (3), so $T(X, Y) = JX \wedge JY + 2\langle JX, Y \rangle J$, where $J = J(x)$ is smooth on $\overline{B} \setminus \{x_0\}$ and has a limit when $x \rightarrow x_0$, $x \in B$ (see the proof of Lemma 9). Using the covariant derivative of T computed in Lemma 8 and (36d), we obtain that on B ,

$$\begin{aligned} (\nabla_U \nabla_Z W)(X, Y) &= \varepsilon \langle H(f)U, Z \rangle (-\tfrac{3}{5}X \wedge Y + T(X, Y)) \\ &+ \tfrac{1}{2}\varepsilon([T(X, Y), H(f)U \wedge Z] + T((H(f)U \wedge Z)X, Y) + T(X, (H(f)U \wedge Z)Y)) \\ &+ \tfrac{1}{2}\varepsilon f^{-1}Zf([T(X, Y), \nabla f \wedge U] + T((\nabla f \wedge U)X, Y) + T(X, (\nabla f \wedge U)Y)) \\ &+ \tfrac{1}{4}\varepsilon f^{-1}[[T(X, Y), \nabla f \wedge U] + T((\nabla f \wedge U)X, Y) + T(X, (\nabla f \wedge U)Y), \nabla f \wedge Z] \\ &+ \tfrac{1}{4}\varepsilon f^{-1}([T((\nabla f \wedge Z)X, Y), \nabla f \wedge U] + T((\nabla f \wedge U)(\nabla f \wedge Z)X, Y) + T((\nabla f \wedge Z)X, (\nabla f \wedge U)Y)) \\ &+ \tfrac{1}{4}\varepsilon f^{-1}([T(X, (\nabla f \wedge Z)Y), \nabla f \wedge U] + T((\nabla f \wedge U)X, (\nabla f \wedge Z)Y) + T(X, (\nabla f \wedge U)(\nabla f \wedge Z)Y)), \end{aligned}$$

where $H(f)$ is the symmetric operator associated to the Hessian of f . Taking $U = Z = E_j$, where $\{E_j\}$ is an orthonormal basis, and summing up by j we find after some computations:

$$\begin{aligned} \sum_{j=1}^6 (\nabla_{E_j} \nabla_{E_j} W)(X, Y) &= \varepsilon \Delta f (-\tfrac{3}{5}X \wedge Y + T(X, Y)) - \varepsilon f^{-1} \|\nabla f\|^2 T(X, Y) \\ &+ \varepsilon f^{-1} (T(X, Y) \nabla f \wedge \nabla f + T((X \wedge Y) \nabla f, \nabla f)) + \tfrac{3}{2} \varepsilon f^{-1} (\nabla f \wedge (X \wedge Y) \nabla f + J \nabla f \wedge (X \wedge Y) J \nabla f). \end{aligned}$$

As both ∇f and J are smooth on $\overline{B} \setminus \{x_0\}$ and have limits when $x \rightarrow x_0$, $x \in B$, there exist unit vector fields X, Y , continuous on \overline{B} and satisfying $\mathcal{I}X, \mathcal{I}Y \perp \nabla f$, $\mathcal{I}X \perp \mathcal{I}Y$. For such X and Y ,

$$\sum_{j=1}^6 (\nabla_{E_j} \nabla_{E_j} W)(X, Y) = \varepsilon \Delta f (-\frac{3}{5} X \wedge Y + JX \wedge JY) - \varepsilon f^{-1} \|\nabla f\|^2 JX \wedge JY.$$

As the left-hand side is continuous on \overline{B} and $\lim_{x \rightarrow x_0, x \in B} \Delta f = 0$ by (37) and Lemma 9, we obtain that the field $f^{-1} \|\nabla f\|^2 JX \wedge JY$ of skew-symmetric operators has a limit at x_0 . Taking the trace of its square we find that there exists a limit $\lim_{x \rightarrow x_0, x \in B} f^{-2} \|\nabla f\|^4$ which implies $\lim_{x \rightarrow x_0, x \in B} \nabla f = 0$ by Lemma 9. We again arrive at a contradiction with the boundary point theorem for the function $u = f$ satisfying (37). \square

REFERENCES

- [ABS] Atiah M.F., Bott R., Shapiro A. *Clifford modules*, Topology, **3**, **suppl.1** (1964), 3 – 38.
- [BG1] Blažić N., Gilkey P. *Conformally Osserman manifolds and conformally complex space forms*, Int. J. Geom. Methods Mod. Phys. **1** (2004), 97 – 106.
- [BG2] Blažić N., Gilkey P. *Conformally Osserman manifolds and self-duality in Riemannian geometry*. Differential geometry and its applications, 15–18, Matfyzpress, Prague, 2005.
- [BGNSi] Blažić N., Gilkey P., Nikčević S., Simon U. *The spectral geometry of the Weyl conformal tensor*. PDEs, submanifolds and affine differential geometry, 195–203, Banach Center Publ., **69**, Polish Acad. Sci., Warsaw, 2005.
- [BGNSt] Blažić N., Gilkey P., Nikčević S., Stavrov I. *Curvature structure of self-dual 4-manifolds*, arXiv: math.DG/0808.2799.
- [Chi] Chi Q.-S. *A curvature characterization of certain locally rank-one symmetric spaces*, J. Differ. Geom. **28**(1988), 187 – 202.
- [DS] Derdzinski A., Shen C.-L. *Codazzi tensor fields, curvature and Pontryagin forms*, Proc. London Math. Soc. (3), **47** (1983), 15 – 26.
- [F] Fraenkel L. E. *An introduction to maximum principles and symmetry in elliptic problems*, Cambridge Tracts in Mathematics, 128. Cambridge University Press, Cambridge, 2000.
- [GKV] Garca-Río E., Kupeli D., Vázquez-Lorenzo R. *Osserman manifolds in semi-Riemannian geometry*. Lecture Notes in Mathematics, 1777. Springer-Verlag, Berlin, 2002.
- [GSV] Gilkey P., Swann A., Vanhecke L. *Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator*, Quart. J. Math. Oxford (2), **46**(1995), 299 – 320.
- [G1] Gilkey P. *Geometric properties of natural operators defined by the Riemann curvature tensor*. World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
- [G2] Gilkey P. *The geometry of curvature homogeneous pseudo-Riemannian manifolds*. ICP Advanced Texts in Mathematics, 2. Imperial College Press, London, 2007.
- [HL] Harvey R., Lawson H.B. *Calibrated geometries*, Acta Math. **148** (1982), 47–157.
- [H] D.Husemoller, *Fiber bundles*, (1975), Springer-Verlag.
- [LM] H.B.Lawson, M.-L.Michelsohn, *Spin geometry*, (1989), Princeton Univ. Press.
- [N1] Nikolayevsky Y. *Osserman manifolds and Clifford structures*, Houston J. Math. **29**(2003), 59–75.
- [N2] Nikolayevsky Y. *Osserman manifolds of dimension 8*, Manuscripta Math. **115**(2004), 31 – 53.
- [N3] Nikolayevsky Y. *Osserman Conjecture in dimension $n \neq 8, 16$* , Math. Ann. **331**(2005), 505 – 522.
- [N4] Nikolayevsky Y. *On Osserman manifolds of dimension 16*, Contemporary Geometry and Related Topics, Proc. Conf. Belgrade, 2005 (2006), 379 – 398.
- [Ol] Olszak Z. *On the existence of generalized space forms*, Israel J. Math. **65**(1989), 214 – 218.
- [Os] Osserman R. *Curvature in the eighties*, Amer. Math. Monthly, **97**(1990), 731 – 756.
- [Pf] Pfister A. *Quadratic forms with applications to algebraic geometry and topology*, London Math. Soc. Lecture Notes Ser., **217**, (1995), Cambridge Univ. Press.

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